

# Massive Gravity as a Quantum Gauge Theory

D. R. Grigore, <sup>1</sup>

Dept. of Theor. Phys., Inst. Atomic Phys.  
Bucharest-Măgurele, P. O. Box MG 6, ROMÂNIA

G. Scharf, <sup>2</sup>

Inst. of Theor. Phys., Univ Zürich.  
Wintherhurerstrs 190, Zürich, SWITZERLAND

## Abstract

We present a new point of view on the quantization of the massive gravitational field, namely we use exclusively the quantum framework of the second quantization. The Hilbert space of the many-gravitons system is a Fock space  $\mathcal{F}^+(\mathbf{H}_{\text{graviton}})$  where the one-particle Hilbert space  $\mathbf{H}_{\text{graviton}}$  carries the direct sum of two unitary irreducible representations of the Poincaré group corresponding to two particles of mass  $m > 0$  and spins 2 and 0, respectively. This Hilbert space is canonically isomorphic to a space of the type  $\text{Ker}(Q)/\text{Im}(Q)$  where  $Q$  is a gauge charge defined in an extension of the Hilbert space  $\mathcal{H}_{\text{graviton}}$  generated by the gravitational field  $h_{\mu\nu}$  and some ghosts fields  $u_\mu, \tilde{u}_\mu$  (which are vector Fermi fields) and  $v_\mu$  (which are vector field Bose fields.)

Then we study the self interaction of massive gravity in the causal framework. We obtain a solution which goes smoothly to the zero-mass solution of linear quantum gravity up to a term depending on the bosonic ghost field. This solution depends on two real constants as it should be; these constants are related to the gravitational constant and the cosmological constant. In the second order of the perturbation theory we do not need a Higgs field, in sharp contrast to Yang-Mills theory.

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<sup>1</sup>e-mail: grigore@theor1.theory.nipne.ro, grigore@theory.nipne.ro

<sup>2</sup>e-mail: scharf@physik.unizh.ch

# 1 Introduction

The quantization of gravity is an old standing problem of quantum field theory. The solution of this problem in full generality is a highly non-trivial problem which seems to be extremely complicated. (See however the papers of Ashtekar and collaborators [1], [32]). In [13] and [14]-[17], [29] this problem was addressed for the linear gravitational field of zero mass. Among the pioneering works in this approach we mention [33], [8], [26], [19], [23], [24]. Using the result of this analysis many computations have been done in the literature (see [3], [20], [4], [36], [35]).

One possible way to perform the quantization of the asymptotic gravitational field is to linearize the classical theory of gravitation using the so called Goldberg variables [10], [18] and then to apply straightforward quantization of the resulting free field theory. Because of the gauge invariance of the theory (which in this case is the invariance under general coordinates transformations) one obtains a constrained system and one tries to use a Bleuler-Gupta type formalism, that is to start with a Hilbert space endowed with a sesquilinear non-degenerate form and select the physical states as a subspace of the type  $Q_A \Phi = 0$ ,  $A = 1, \dots, N$ .

A related idea is to extend the Fock space to an auxiliary Hilbert space  $\mathcal{H}^{gh}$  including some fictitious fields, called ghosts, and construct a gauge charge (i.e. an operator  $Q$  verifying  $Q^2 = 0$ ) such that the physical Hilbert space is  $\mathcal{H}_{phys} \equiv Ker(Q)/Im(Q)$  (see for instance [24] and references quoted there). As a result of this procedure, it is asserted that the *graviton*, i.e. the elementary quantum particle must be a massless spin 2 particle. The construction of the gauge charge relies heavily on classical field theory arguments, because one tries to obtain for the quantum gauge transformations expressions of the same type as the general coordinates transformations appearing in general relativity. This invariance is then promoted to a quantum gauge invariance which should be implemented by the commutator with the gauge charge  $Q$ .

It is an interesting problem to consider the case of massive gravity. This case was analyzed many times ago [5], [34]. In [5] it is argued that even the quantization of the massive spin 2 field is problematic in the sense that no smooth limit  $m \mapsto 0$  exists. Some recent interest on this problem exists [2], [25], [27], [21] and [6].

We will show here that one can perform the quantization in such a way that this limit is smooth. One finds out that the massive graviton has a scalar partner of the same mass  $m$ . The construction is done in the spirit of [13].

We also mention that a rigorous construction of the Hilbert space of the many-gravitons system is indispensable for the construction of the corresponding  $S$ -matrix in perturbation theory in the sense of Bogoliubov. This construction emphasizes the basic rôle of causality in quantum field theory. We obtain a solution for the interaction Lagrangian (the first-order chronological product) which goes smoothly for  $m \searrow 0$  into the solution appearing in [29].

The solution we obtain, up to second order of the perturbation theory, coincides with the result of the perturbative development of the Einstein-Hilbert Lagrangian with cosmological constant, if we make the identification  $\Lambda = 2m^2$  and use Goldberg variables (see the Conclusions). We remark that in the second order of the perturbation theory we do not need a Higgs field as in the case of Yang-Mills theory. For this reason it seems to be impossible to find our massive spin 2 gauge theory by means of the conventional Higgs mechanism.

## 2 The Quantization of the Asymptotic Massive Gravitational Field

One defines the graviton as a certain unitary irreducible representation of the Poincaré group corresponding to zero mass and helicity 2. In the case of massive gravity one should use the representation of positive mass  $m$  and spin 2. These representation can be explicitly described using the formalism of Hilbert space bundles, as presented for instance in [34], ch. VI.7 thm 6.20. Let us denote by  $H_{\text{gr}}^{(m)}$  the one-particle Hilbert space of the graviton of mass  $m$ . The ensemble of many gravitons is usually described by the associated Fock space  $\mathcal{F}_{\text{graviton}} = \mathcal{F}^{(+)}(H_{\text{gr}}^{(m)})$  where the upper  $+$  sign indicates that the gravitons are assumed to be Bosons according the the well-known spin-statistics theorem. The Hilbert space  $\mathcal{F}_{\text{graviton}}$  is not very suitable for the construction of the perturbative series of the scattering matrix  $S$  in the sense of Bogoliubov. The way out is to construct a larger Hilbert space  $\mathcal{H}$  where unphysical degrees of freedom are present. In this Hilbert space a (gravitational) gauge charge  $Q$  acts which should be chosen such that it squares to zero  $Q^2 = 0$ ; in this case it makes sense to consider the factor space  $\mathcal{H}_{\text{phys}} \equiv \text{Ker}(Q)/\text{Im}(Q)$  which should be canonically isomorphic to  $\mathcal{F}_{\text{graviton}}$ .

Let us describe this construction. We use in this paper the following notations. The upper hyperboloid of mass  $m \geq 0$  is by definition  $X_m^+ \equiv \{p \in \mathbb{R}^4 \mid \|p\|^2 = m^2\}$ ; it is a Borel set with the Lorentz invariant measure  $d\alpha_m^+(p) \equiv \frac{d\mathbf{p}}{2\omega(\mathbf{p})}$ . Here the conventions are the following:  $\|\cdot\|$  is the Minkowski norm defined by  $\|p\|^2 \equiv p \cdot p$  and  $p \cdot q$  is the Minkowski bilinear form:  $p \cdot q \equiv p_0 q_0 - \mathbf{p} \cdot \mathbf{q}$ ; by  $\eta_{\mu\nu}$  we denote the corresponding flat Minkowski matrix with diagonal elements 1, -1, -1, -1. If  $\mathbf{p} \in \mathbb{R}^3$  we define  $\tau(\mathbf{p}) \in X_m^+$  according to  $\tau(\mathbf{p}) \equiv (\omega(\mathbf{p}), \mathbf{p})$ ,  $\omega(\mathbf{p}) \equiv \sqrt{\mathbf{p}^2 + m^2}$ .

First we consider the zero mass case  $m = 0$  [13], [29].

- One generates the Hilbert space  $\mathcal{H}$  by applying on the vacuum the fields  $H_{\mu\nu}, u_\rho, \tilde{u}_\rho, \Phi$  (the rigorous construction is based on the Borchers algebra); these fields are of null mass:

$$\partial^2 H_{\mu\nu}(x) = 0 \quad \partial^2 u_\rho(x) = 0 \quad \partial^2 \tilde{u}_\rho(x) = 0 \quad \partial^2 \Phi(x) = 0 \quad (2.1)$$

- $H_{\mu\nu}$  is symmetric and traceless:

$$H_{\mu\nu} = H_{\nu\mu} \quad H^\mu{}_\mu = 0 \quad (2.2)$$

- The field  $\Phi$  is scalar and  $H_{\mu\nu}, u_\rho, \tilde{u}_\rho$  have obvious tensor and vector properties
- The fields  $H_{\mu\nu}, \Phi$  are Bosons and  $u_\rho, \tilde{u}_\rho$  are Fermions
- The causal commutation relations of these fields are

$$\begin{aligned} [H_{\rho\sigma}(x), H_{\lambda\omega}(y)] &= -\frac{i}{2} \left( \eta_{\rho\lambda}\eta_{\sigma\omega} + \eta_{\rho\omega}\eta_{\sigma\lambda} - \frac{1}{2}\eta_{\rho\sigma}\eta_{\lambda\omega} \right) D_0(x-y) \times \mathbf{1} \\ \{u_\mu(x), \tilde{u}_\nu(y)\} &= i \eta_{\mu\nu} D_0(x-y) \mathbf{1} \\ [\Phi(x), \Phi(y)] &= i D_0(x-y) \mathbf{1} \end{aligned} \quad (2.3)$$

and the other (anti)commutators are zero; in particular all Bose fields commute with all Fermi fields. Here

$$D_m(x) = D_m^{(+)}(x) + D_m^{(-)}(x) \quad (2.4)$$

is the Pauli-Jordan distribution of mass  $m \geq 0$  and  $D_m^{(\pm)}(x)$  are given by:

$$D_m^{(\pm)}(x) \equiv \pm \frac{i}{(2\pi)^{3/2}} \int_{X_m^+} d\alpha_m^+(p) e^{\mp i p \cdot x}. \quad (2.5)$$

- In this Hilbert space there exists a sesqui-linear form (not positively defined)  $\langle \cdot, \cdot \rangle$  such that we have

$$H_{\mu\nu}^\dagger = H_{\mu\nu} \quad u_\mu^\dagger = u_\mu \quad \tilde{u}_\mu^\dagger = -\tilde{u}_\mu \quad \Phi^\dagger = \Phi \quad (2.6)$$

where by  $\dagger$  we mean the adjoint with respect to  $\langle \cdot, \cdot \rangle$

- The operator  $Q$  is well defined through the relations

$$Q\Omega = 0 \quad (2.7)$$

$$\begin{aligned} [Q, H_{\mu\nu}] &= -\frac{i}{2} \left( \partial_\mu u_\nu + \partial_\nu u_\mu - \frac{1}{2} \eta_{\mu\nu} \partial_\rho u_\sigma \right) & [Q, \Phi] &= \frac{i}{2} \partial^\rho u_\rho \\ \{Q, u_\mu\} &= 0, & \{Q, \tilde{u}_\mu\} &= i \left( \partial^\nu H_{\mu\nu} + \frac{1}{2} \partial_\mu \Phi \right). \end{aligned} \quad (2.8)$$

In these conditions one can prove that:

- The operator  $Q$  is well defined; for this one has to check the validity of the Jacoby identity:

$$[b(x), \{f(y), Q\}] + \{f(y), [Q, b(x)]\} = 0 \quad (2.9)$$

where  $b$  and  $f$  are generic Bose (resp. Fermi) fields.

- The following relations are verified:

$$Q^2 = 0 \quad (2.10)$$

$$\mathcal{U}_g Q = Q \mathcal{U}_g, \quad \forall g \in \mathcal{P}. \quad (2.11)$$

From (2.10) we have

$$Im(Q) \subset Ker(Q) \quad (2.12)$$

so it makes sense to consider the factor space  $Ker(Q)/Im(Q)$ . One can prove that the sesqui-linear form  $\langle \cdot, \cdot \rangle$  induces a strictly positively defined scalar product on  $\overline{Ker(Q)/Im(Q)}$  and we have a canonical isomorphism  $\overline{Ker(Q)/Im(Q)} \sim \mathcal{F}_{gravitation}$ .

The preceding construction presented in detail in [13] justifies the consideration of the auxiliary Hilbert space  $\mathcal{H}$  as a lay-ground for the perturbation theory. The fields  $u_\rho, \tilde{u}_\rho, \Phi$  are called *ghost* fields and the operator  $Q$  is the *gauge charge*. A simplification of the preceding formalism is the consideration of the new field

$$h_{\mu\nu} \equiv H_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Phi \quad (2.13)$$

which is self-adjoint in the sense

$$h_{\mu\nu}(x)^\dagger = h_{\mu\nu}(x) \quad (2.14)$$

but is not traceless anymore and the causal commutation relations are:

$$[h_{\rho\sigma}(x), h_{\lambda\omega}(y)] = -\frac{i}{2}(\eta_{\rho\lambda}\eta_{\sigma\omega} + \eta_{\rho\omega}\eta_{\sigma\lambda} - \eta_{\rho\sigma}\eta_{\lambda\omega}) D_0(x-y) \times \mathbf{1}. \quad (2.15)$$

We can easily prove that the preceding definition of the gauge charge is equivalently described by (2.7) and:

$$\begin{aligned} [Q, h_{\mu\nu}] &= -\frac{i}{2}(\partial_\mu u_\nu + \partial_\nu u_\mu - \eta_{\mu\nu}\partial_\rho u^\rho) \\ \{Q, u_\mu\} &= 0, \quad \{Q, \tilde{u}_\mu\} = i \partial^\nu h_{\mu\nu} \end{aligned} \quad (2.16)$$

so one can consider that the Hilbert space  $\mathcal{H}$  is generated by the fields  $h_{\mu\nu}, u_\rho, \tilde{u}_\rho$  with the properties described above.

We now turn to the massive gravitational field. One notices from the very beginning that in the case  $m > 0$  the gauge charge defined by (2.16) does not square to zero anymore. One can try to correct this feature as in the case of the massive vector field (see for instance [29]) by introducing a new ghost field  $v_\mu$  which is a vector field. The one modifies the preceding scheme as follows:

- One generates the Hilbert space  $\mathcal{H}$  by applying on the vacuum the fields  $h_{\mu\nu}, u_\rho, \tilde{u}_\rho, v_\mu$ ; all these fields are of mass  $m$ :

$$(\partial^2 + m^2)h_{\mu\nu}(x) = 0 \quad (\partial^2 + m^2)u_\rho(x) = 0 \quad (\partial^2 + m^2)\tilde{u}_\rho(x) = 0 \quad (\partial^2 + m^2)v_\mu(x) = 0 \quad (2.17)$$

- $h_{\mu\nu}$  is symmetric:

$$h_{\mu\nu} = h_{\nu\mu} \quad (2.18)$$

- The fields  $h_{\mu\nu}, u_\rho, \tilde{u}_\rho, v_\mu$  have obvious tensor and vector properties
- The fields  $h_{\mu\nu}, v_\mu$  are Bosons and  $u_\rho, \tilde{u}_\rho$  are Fermions
- The causal commutation relations of these fields are

$$\begin{aligned} [h_{\rho\sigma}(x), h_{\lambda\omega}(y)] &= -\frac{i}{2}(\eta_{\rho\lambda}\eta_{\sigma\omega} + \eta_{\rho\omega}\eta_{\sigma\lambda} - \eta_{\rho\sigma}\eta_{\lambda\omega}) D_m(x-y) \times \mathbf{1} \\ \{u_\mu(x), \tilde{u}_\nu(y)\} &= i \eta_{\mu\nu} D_m(x-y) \mathbf{1} \\ [v_\mu(x), v_\rho(y)] &= \frac{i}{2} D_m(x-y) \mathbf{1} \end{aligned} \quad (2.19)$$

and the other (anti)commutators are zero

- In this Hilbert space there exists a sesqui-linear form (not positively defined)  $\langle \cdot, \cdot \rangle$  such that we have

$$h_{\mu\nu}^\dagger = h_{\mu\nu} \quad u_\mu^\dagger = u_\mu \quad \tilde{u}_\mu^\dagger = -\tilde{u}_\mu \quad v_\mu^\dagger = v_\mu \quad (2.20)$$

where by  $\dagger$  we mean the adjoint with respect to  $\langle \cdot, \cdot \rangle$

- The operator  $Q$  is well defined through the relations(2.7 ) and

$$\begin{aligned} [Q, h_{\mu\nu}] &= -\frac{i}{2} (\partial_\mu u_\nu + \partial_\nu u_\mu - \eta_{\mu\nu} \partial_\rho u_\sigma) \quad [Q, v_\mu] = -\frac{im}{2} u_\mu \\ \{Q, u_\mu\} &= 0, \quad \{Q, \tilde{u}_\mu\} = i (\partial^\nu h_{\mu\nu} + m v_\mu). \end{aligned} \quad (2.21)$$

In these conditions one can prove that the operator  $Q$  is well defined because of the validity of the Jacobi identity (2.9) and we also have (2.10) and (2.11), so again it makes sense to consider the factor space  $\overline{Ker(Q)/Im(Q)}$ . One can prove in this case also that the sesqui-linear form  $\langle \cdot, \cdot \rangle$  induces a strictly positively defined scalar product on this factor space. However, in this case a modification of the zero mass scheme appears. The one-particle Hilbert corresponding to  $\overline{Ker(Q)/Im(Q)}$  is  $H^{[m,2]} \oplus H^{[m,0]}$  i.e. it describes two particles of mass  $m$ , one of spins 2 and one of spin 0, respectively. In other words we have  $\overline{Ker(Q)/Im(Q)} = \mathcal{F}_{graviton} \oplus \mathcal{F}_{scalar}$ . It seems impossible to construct a gauge structure such that the scalar partner of the graviton is eliminated, so in this paper we will accept that such a particle does exists. It remains to be investigated whether the scalar partner of the graviton with a tiny mass leads to phenomenological problems.

Sometimes it is convenient to generalize the expression of the new field (2.13) in the sense:

$$h_{\mu\nu}^{(\alpha)} \equiv H_{\mu\nu} + \frac{1}{2}\alpha \eta_{\mu\nu} \Phi \quad (2.22)$$

with  $\alpha \in \mathbb{R}^*$ . The causal commutation relations are for this field:

$$\left[ h_{\rho\sigma}^{(\alpha)}(x), h_{\lambda\omega}^{(\alpha)}(y) \right] = -\frac{i}{2} \left( \eta_{\rho\lambda} \eta_{\sigma\omega} + \eta_{\rho\omega} \eta_{\sigma\lambda} - \frac{1+\alpha^2}{2} \eta_{\rho\sigma} \eta_{\lambda\omega} \right) D_m(x-y) \times \mathbf{1}. \quad (2.23)$$

We can prove that the definition of the gauge charge is equivalently described by (2.7) and:

$$[Q, h_{\mu\nu}^{(\alpha)}] = -\frac{i}{2} \left( \partial_\mu u_\nu + \partial_\nu u_\mu - \frac{1+\alpha}{2} \eta_{\mu\nu} \partial_\rho u^\rho \right), \quad (2.24)$$

$$\{Q, u_\mu\} = 0, \quad \{Q, \tilde{u}_\mu\} = i \left( \partial^\nu h_{\mu\nu}^{(\alpha)} + \frac{1-\alpha}{4\alpha} \partial_\mu h^{(\alpha)} + m v_\mu \right) \quad (2.25)$$

and

$$[Q, v_\mu] = -\frac{im}{2} u_\mu; \quad (2.26)$$

here

$$h^{(\alpha)} \equiv \eta^{\mu\nu} h_{\mu\nu}^{(\alpha)}. \quad (2.27)$$

The choice (2.13) correspond to  $\alpha = 1$ . Let us consider the choice  $\alpha = -1$ . Then the preceding relations for

$$\hat{h}_{\mu\nu} = h_{\mu\nu}^{(-1)} \quad (2.28)$$

become:

$$\left[ \hat{h}_{\rho\sigma}(x), \hat{h}_{\lambda\omega}(y) \right] = -\frac{i}{2} (\eta_{\rho\lambda}\eta_{\sigma\omega} + \eta_{\rho\omega}\eta_{\sigma\lambda} - \eta_{\rho\sigma}\eta_{\lambda\omega}) D_m(x-y) \times \mathbf{1}. \quad (2.29)$$

$$[Q, \hat{h}_{\mu\nu}] = -\frac{i}{2} (\partial_\mu u_\nu + \partial_\nu u_\mu), \quad (2.30)$$

$$\{Q, u_\mu\} = 0, \quad \{Q, \tilde{u}_\mu\} = i \left( \partial^\nu \hat{h}_{\mu\nu} - \frac{1}{2} \partial_\mu \hat{h} + m v_\mu \right) \quad (2.31)$$

and

$$[Q, v_\mu] = -\frac{im}{2} u_\mu. \quad (2.32)$$

This choice seems to appear naturally in the classical framework of gravity with an non-zero cosmological constant, if one expands the metric  $g_{\mu\nu}$  around Minkowski background in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa \hat{h}_{\mu\nu}$$

(see the Conclusions). However, from the quantum point of view the value of  $\alpha$  is irrelevant: all choices are good for the description of the physical Hilbert space.

We remark also that the massless limit problem mentioned in [5] has a very simple explanation according to the preceding observation: in [5] one uses different values of the parameter  $\alpha$  for the case  $m = 0$  and  $m > 0$  respectively. The correct procedure is to use the same value of  $\alpha$  in both cases.

The construction of observables can be done in the usual way. We denote by  $\mathcal{W}$  the linear space of all Wick monomials on the Fock space  $\mathcal{H}^{gh}$  i.e. containing the fields  $h_{\mu\nu}(x)$ ,  $u_\mu(x)$ ,  $\tilde{u}_\mu(x)$  and  $v_\mu(x)$ . If  $M$  is such a Wick monomial, we define by  $gh_\pm(M)$  the degree in  $\tilde{u}_\mu$  (resp. in  $u_\mu$ ). The total degree of  $M$  is

$$deg(M) \equiv gh_+(M) + gh_-(M). \quad (2.33)$$

The *ghost number* is, by definition, the expression:

$$gh(M) \equiv gh_+(M) - gh_-(M). \quad (2.34)$$

If  $M \in \mathcal{W}$  let us define the operator:

$$d_Q M \equiv: QM : - (-1)^{gh(M)} : MQ : \quad (2.35)$$

on monomials  $M$  and extend it by linearity to the whole  $\mathcal{W}$ . Then  $d_Q M \in \mathcal{W}$  and

$$gh(d_Q M) = gh(M) - 1. \quad (2.36)$$

The operator  $d_Q : \mathcal{W} \rightarrow \mathcal{W}$  is called the gauge variation; the properties of this object are summarized in the following relations:

$$d_Q^2 = 0 \quad (2.37)$$

$$\begin{aligned}
d_Q h_{\mu\nu} &= -\frac{i}{2} (\partial_\mu u_\nu + \partial_\nu u_\mu - \eta_{\mu\nu} \partial_\rho u^\rho) \\
d_Q u_\mu &= 0 \quad d_Q \tilde{u}_\mu = i (\partial^\nu h_{\mu\nu} + m v_\mu) \\
d_Q v_\mu &= -\frac{im}{2} u_\mu(x).
\end{aligned} \tag{2.38}$$

$$d_Q(MN) = (d_Q M)N + (-1)^{gh(M)} M(d_Q N), \quad \forall M, N \in \mathcal{W}. \tag{2.39}$$

If  $O : \mathcal{H}^{gh} \rightarrow \mathcal{H}^{gh}$  verifies the condition

$$d_Q O = 0 \tag{2.40}$$

then it induces a well defined operator  $[O]$  on the factor space  $\overline{Ker(Q)}/\overline{Im(Q)}$ .

Moreover, in this case the following formula is true for the matrix elements of the factorized operator  $[O]$ :

$$([\Psi], [O][\Phi]) = (\Psi, O\Phi). \tag{2.41}$$

This kind of observables on the physical space will also be called *gauge invariant observables*. An operator  $O : \mathcal{H}^{gh} \rightarrow \mathcal{H}^{gh}$  induces a gauge invariant observables if and only if it verifies:

$$d_Q O|_{Ker(Q)} = 0. \tag{2.42}$$

Not all operators verifying the condition (2.40) are interesting. In fact, the operators of the type  $d_Q O$  are inducing a null operator on the factor space; explicitly we have:

$$[d_Q O] = 0. \tag{2.43}$$

In the framework of perturbative quantum field theory the axiom of factorization in the adiabatic limit is: XX

$$\lim_{\epsilon \searrow 0} d_Q \int_{\mathbb{R}^4} dx T_n(x_1, \dots, x_n) g(\epsilon x)|_{Ker(Q)} = 0, \quad \forall n \in \mathbb{N}^*. \tag{2.44}$$

If infrared divergences cannot be avoided, the one can consider the preceding relation at the heuristic level and impose the postulate:

$$d_Q T_n(x_1, \dots, x_n) = i \sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} T_{n/l}^\mu(x_1, \dots, x_n), \quad \forall n \in \mathbb{N}^* \tag{2.45}$$

as it is done in [28]. In particular we have for  $n = 1$

$$d_Q T_1(x) = i \frac{\partial}{\partial x^\mu} T^\mu(x) \tag{2.46}$$

for some Wick polynomials  $T^\mu(x)$ . The derivation of the most general expression of  $T_1$  can be done in the original variables  $H_{\mu\nu}, \Phi, u_\mu, \tilde{u}_\mu, v_\mu$  or with  $h_{\mu\nu}^{(\alpha)}$  for any values of  $\alpha$ . If we change the fields, we must correspondingly change the expression of the operator  $d_Q$  and the final result should be the same. In formulae:

$$d_Q T_1(H_{\mu\nu}, \dots) = i \frac{\partial}{\partial x^\mu} T^\mu(H_{\mu\nu}, \dots) \Leftrightarrow d_Q T_1(h_{\mu\nu}^{(\alpha)}, \dots) = i \frac{\partial}{\partial x^\mu} T^\mu(h_{\mu\nu}^{(\alpha)}, \dots) \tag{2.47}$$

for any  $\alpha$ .



### 3 First Order Gauge Invariance

In view of the discussion from the preceding Section it is natural to discard from the interaction Lagrangian (the first-order chronological product  $T_1$ ) expressions of the type

$$d_Q B + \partial_\mu B^\mu, \quad gh(B) = -1, gh(B^\mu) = 0; \quad (3.1)$$

we call such an expression a *trivial coupling*. If the difference of two couplings is a trivial one then we call them *equivalent*. In this Section we prove the following

**Theorem 3.1** *Let us consider the most general Wick polynomial  $T$  tri-linear in the fields  $H_{\mu\nu}, \Phi, u_\mu, \tilde{u}_\mu, v_\mu$  verifying the following conditions:*

$$\begin{aligned} U_g T &= T U_g, \quad \forall g \in \mathcal{P} \\ gh(T) &= 0 \\ 3 &\leq \deg(T) \leq 5 \end{aligned} \quad (3.2)$$

and the gauge invariance condition (2.46). Then  $T$  is equivalent to the following expression:

$$T = a T^{(a)} + b T^{(b)} \quad a, b \in \mathbb{R} \quad (3.3)$$

where

$$\begin{aligned} T^{(a)} \equiv & [-2H^{\mu\nu}(\partial_\mu H_{\rho\sigma})(\partial_\nu H^{\rho\sigma}) - 4H^{\mu\nu}(\partial^\alpha H^{\rho\mu})(\partial^\rho H_{\alpha\nu}) + 2\Phi(\partial^\mu H^{\rho\sigma})(\partial_\rho H_{\mu\sigma}) \\ & + 4\Phi H^{\rho\sigma}(\partial_\rho \partial^\nu H_{\nu\sigma}) + \Phi^2(\partial_\mu \partial_\nu H^{\mu\nu}) + (\partial^\mu \Phi)(\partial^\nu \Phi)H_{\mu\nu} + 4H^{\mu\nu}u^\rho(\partial_\mu \partial_\nu \tilde{u}_\rho) \\ & + 4(\partial_\mu H^{\mu\nu})u^\rho(\partial_\nu \tilde{u}_\rho) + 4(\partial_\mu H^{\mu\nu})u^\rho(\partial_\rho \tilde{u}_\nu) + 2(\partial^\mu \Phi)u^\rho(\partial_\rho \tilde{u}_\mu) + 4H^{\mu\nu}(\partial_\mu v_\rho)(\partial_\nu v^\rho) \\ & - 4m(\partial_\mu v_\nu)u^\mu \tilde{u}^\nu + m^2 \left( \frac{2}{3}H^{\mu\nu}H_{\mu\rho}H_\nu{}^\rho + \frac{1}{2}H^{\mu\nu}H_{\mu\nu}\Phi - \frac{3}{4}\Phi^3 - \Phi u^\mu \tilde{u}_\mu + \Phi v^\mu v_\mu \right) \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} T^{(b)} \equiv & -H^{\mu\nu}(\partial_\mu v_\rho)(\partial_\nu v^\rho) + 2H^{\mu\nu}(\partial_\mu v_\nu)(\partial_\rho v^\rho) - 2H^{\mu\nu}(\partial_\mu v_\rho)(\partial^\rho v^\nu) \\ & + m[H^{\mu\nu}(\partial_\rho H_{\mu\nu})v^\rho - 2H^{\mu\nu}(\partial_\nu H_{\mu\rho})v^\rho + H^{\mu\nu}(\partial_\mu \Phi)v_\nu + (\partial_\rho v^\rho)u_\mu \tilde{u}^\mu - \frac{1}{2}(\partial_\mu v_\nu)u^\mu \tilde{u}^\nu \\ & + (\partial_\nu v_\mu)u^\mu \tilde{u}^\nu + v^\nu u^\mu(\partial_\nu \tilde{u}_\mu) + v_\mu u^\mu(\partial^\nu \tilde{u}_\mu) - \frac{1}{2}v^\nu u^\mu(\partial_\mu \tilde{u}_\nu) + 2v_\mu v_\nu(\partial^\mu v^\nu) \\ & + m^2 \left( -\frac{1}{3}H^{\mu\nu}H_{\mu\rho}H_\nu{}^\rho - H^{\mu\nu}u_\mu \tilde{u}_\nu + \frac{3}{2}H^{\mu\nu}v_\mu v_\nu \right). \end{aligned} \quad (3.5)$$

The preceding expression has a smooth limit  $m \rightarrow 0$ .

**Proof:** We make a list of all Wick monomials verifying the conditions (3.2). The condition of Lorentz invariance depends essentially on the dimension of the space-time. In other dimensions than 4 the list below changes drastically. Also the fact that  $H_{\mu\nu}$  is traceless is useful in eliminating many terms. First we have terms which do also appear for the massless case i.e. without the ghost field  $v_\mu$  namely:

$$\begin{aligned}
T^{(1)} &= c^{(1)} H^{\mu\nu} H_{\mu\rho} H_{\nu}^{\rho} \\
T^{(2)} &= c^{(2)} H^{\mu\nu} H_{\mu\nu} \Phi \\
T^{(3)} &= c^{(3)} \Phi^3 \\
T^{(4)} &= c^{(4)} H^{\mu\nu} u_{\mu} \tilde{u}_{\nu} \\
T^{(5)} &= c^{(5)} \Phi u^{\mu} \tilde{u}_{\mu} \\
T^{(6)} &= c_1^{(6)} H^{\mu\nu} (\partial_{\mu} H_{\rho\sigma}) (\partial_{\nu} H^{\rho\sigma}) + c_2^{(6)} H^{\mu\nu} (\partial_{\rho} H_{\rho\sigma}) (\partial_{\sigma} H^{\mu\nu}) + c_3^{(6)} H^{\mu\nu} (\partial_{\mu} H^{\rho\sigma}) (\partial_{\rho} H_{\nu\sigma}) \\
&\quad + c_4^{(6)} H^{\mu\nu} (\partial_{\mu} H_{\rho\nu}) (\partial_{\sigma} H^{\sigma\rho}) + c_5^{(6)} H^{\mu\nu} (\partial^{\rho} H_{\rho\mu}) (\partial^{\sigma} H_{\sigma\nu}) + c_6^{(6)} H^{\mu\nu} (\partial^{\alpha} H^{\rho\mu}) (\partial^{\rho} H_{\alpha\nu}) \\
&\quad + c_7^{(6)} \epsilon_{\mu\rho\alpha\lambda} H^{\mu\nu} (\partial^{\lambda} H_{\nu}^{\rho}) (\partial_{\beta} H^{\alpha\beta}) + c_8^{(6)} \epsilon_{\mu\rho\alpha\lambda} H^{\mu\nu} (\partial^{\lambda} H^{\rho\sigma}) (\partial_{\sigma} H_{\nu}^{\alpha}) \\
&\quad + c_9^{(6)} \epsilon_{\mu\rho\alpha\lambda} H^{\mu\nu} (\partial^{\lambda} H^{\rho\sigma}) (\partial_{\nu} H_{\sigma}^{\alpha}) \\
T^{(7)} &= c_1^{(7)} \Phi (\partial_{\mu} H^{\mu\sigma}) (\partial^{\nu} H_{\nu\sigma}) + c_2^{(7)} \Phi (\partial^{\mu} H^{\rho\sigma}) (\partial_{\rho} H_{\mu\sigma}) + c_3^{(7)} \epsilon_{\mu\nu\rho\alpha} \Phi (\partial^{\mu} H^{\rho\sigma}) (\partial^{\nu} H_{\sigma}^{\alpha}) \\
T^{(8)} &= c^{(8)} \Phi H^{\rho\sigma} (\partial_{\rho} \partial^{\nu} H_{\nu\sigma}) \\
T^{(9)} &= c^{(9)} \Phi^2 (\partial_{\mu} \partial_{\nu} H^{\mu\nu}) \\
T^{(10)} &= c^{(10)} (\partial^{\mu} \Phi) (\partial^{\nu} \Phi) H_{\mu\nu} \\
T^{(11)} &= c_1^{(11)} (\partial_{\mu} H^{\mu\nu}) u_{\nu} (\partial_{\rho} \tilde{u}^{\rho}) + c_2^{(11)} (\partial_{\mu} H^{\mu\nu}) u^{\rho} (\partial_{\nu} \tilde{u}_{\rho}) + c_3^{(11)} (\partial_{\mu} H^{\mu\nu}) u^{\rho} (\partial_{\rho} \tilde{u}_{\nu}) \\
&\quad + c_4^{(11)} (\partial^{\rho} H^{\mu\nu}) u_{\mu} (\partial_{\nu} \tilde{u}_{\rho}) + c_5^{(11)} (\partial^{\rho} H^{\mu\nu}) u_{\rho} (\partial_{\mu} \tilde{u}_{\nu}) \\
&\quad + c_6^{(11)} \epsilon_{\mu\rho\alpha\sigma} (\partial^{\rho} H^{\mu\nu}) u^{\alpha} (\partial^{\sigma} \tilde{u}_{\nu}) + c_7^{(11)} \epsilon_{\mu\rho\alpha\beta} (\partial^{\rho} H^{\mu\nu}) u^{\alpha} (\partial_{\nu} \tilde{u}^{\beta}) + c_8^{(11)} \epsilon_{\mu\sigma\alpha\beta} (\partial_{\nu} H^{\mu\nu}) u^{\alpha} (\partial^{\sigma} \tilde{u}^{\beta}) \\
T^{(12)} &= c_1^{(12)} (\partial^{\rho} \partial_{\nu} H^{\mu\nu}) u_{\mu} \tilde{u}_{\rho} + c_2^{(12)} (\partial^{\rho} \partial_{\nu} H^{\mu\nu}) u_{\rho} \tilde{u}_{\mu} + c_3^{(12)} (\partial_{\mu} \partial_{\nu} H^{\mu\nu}) u^{\rho} \tilde{u}_{\rho} \\
&\quad + c_4^{(12)} \epsilon_{\mu\rho\alpha\beta} (\partial^{\rho} \partial_{\nu} H^{\mu\nu}) u^{\alpha} \tilde{u}^{\beta} \\
T^{(13)} &= c_1^{(13)} H^{\mu\nu} u_{\mu} (\partial_{\nu} \partial_{\rho} \tilde{u}_{\rho}) + c_2^{(13)} H^{\mu\nu} u_{\rho} (\partial^{\rho} \partial_{\nu} \tilde{u}_{\mu}) + c_3^{(13)} H^{\mu\nu} u^{\rho} (\partial_{\mu} \partial_{\nu} \tilde{u}_{\rho}) \\
&\quad + c_4^{(13)} \epsilon_{\mu\rho\alpha\beta} H^{\mu\nu} u^{\alpha} (\partial^{\rho} \partial_{\nu} \tilde{u}_{\beta}) \\
T^{(14)} &= c_1^{(14)} (\partial^{\mu} \Phi) u_{\mu} (\partial_{\rho} \tilde{u}^{\rho}) + c_2^{(14)} (\partial^{\mu} \Phi) u^{\rho} (\partial_{\rho} \tilde{u}_{\mu}) + c_3^{(14)} \epsilon_{\mu\nu\alpha\beta} (\partial^{\mu} \Phi) u^{\alpha} (\partial^{\nu} \tilde{u}^{\beta}) \\
T^{(15)} &= c^{(15)} \Phi u_{\mu} (\partial^{\mu} \partial^{\nu} \tilde{u}_{\nu}) \\
T^{(16)} &= c^{(15)} (\partial^{\mu} \partial^{\nu} \Phi) u_{\mu} \tilde{u}_{\nu}; \quad (3.6)
\end{aligned}$$

then we have the terms containing at least one factor  $v_{\mu}$  namely:

$$\begin{aligned}
U^{(1)} &= d^{(1)} H^{\mu\nu} v_{\mu} v_{\nu} \\
U^{(2)} &= d^{(2)} \Phi v^{\mu} v_{\mu} \\
U^{(3)} &= d_1^{(3)} H^{\mu\nu} (\partial_{\alpha} H_{\mu\nu}) A^{\alpha} + d_2^{(3)} H^{\mu\nu} (\partial^{\alpha} H_{\mu\alpha}) v_{\nu} \\
&\quad + d_3^{(3)} H^{\mu\nu} (\partial_{\nu} H_{\mu\alpha}) v^{\alpha} + d_4^{(3)} \epsilon_{\mu\rho\alpha\beta} H^{\mu\nu} (\partial^{\alpha} H_{\nu}^{\rho}) v^{\beta} \\
U^{(4)} &= d^{(4)} H^{\mu\nu} (\partial_{\mu} \Phi) v_{\nu} \\
U^{(5)} &= d^{(5)} \Phi (\partial^{\nu} H_{\mu\nu}) v_{\mu} \\
U^{(6)} &= d^{(6)} \Phi (\partial^{\alpha} \Phi) v_{\alpha} \\
U^{(7)} &= d_1^{(7)} (\partial_{\alpha} v^{\alpha}) u_{\mu} \tilde{u}^{\mu} + d_2^{(7)} (\partial_{\mu} v_{\nu}) u^{\mu} \tilde{u}^{\nu} + d_3^{(7)} (\partial_{\nu} v_{\mu}) u^{\mu} \tilde{u}^{\nu} + d_4^{(7)} \epsilon_{\mu\nu\alpha\beta} (\partial^{\alpha} v^{\beta}) u^{\mu} \tilde{u}^{\nu}
\end{aligned}$$

$$\begin{aligned}
U^{(8)} &= d_1^{(8)} v^\alpha u^\mu (\partial_\alpha \tilde{u}_\mu) + d_2^{(8)} v_\mu u^\mu (\partial_\nu \tilde{u}^\nu) + d_3^{(8)} v^\nu u^\mu (\partial_\mu \tilde{u}_\nu) + d_4^{(8)} \epsilon_{\mu\nu\alpha\beta} v^\alpha u^\mu (\partial^\beta \tilde{u}^\nu) \\
U^{(9)} &= d^{(9)} v_\mu v_\nu (\partial^\mu v^\nu) \\
U^{(10)} &= d_1^{(10)} H^{\mu\nu} (\partial_\mu v_\alpha) (\partial_\nu v^\alpha) + d_2^{(10)} H^{\mu\nu} (\partial_\mu v_\nu) (\partial_\beta v^\beta) + d_3^{(10)} H^{\mu\nu} (\partial_\mu v_\rho) (\partial^\rho v_\nu) \\
&\quad + d_4^{(10)} \epsilon_{\mu\rho\alpha\beta} H^{\mu\nu} (\partial^\rho v_\nu) (\partial^\alpha v^\beta) + d_5^{(10)} \epsilon_{\mu\rho\alpha\beta} H^{\mu\nu} (\partial^\rho v^\alpha) (\partial_\nu v^\beta) \\
U^{(11)} &= d_1^{(11)} H^{\mu\nu} v^\alpha (\partial_\mu \partial_\nu v_\alpha) + d_2^{(11)} H^{\mu\nu} v_\nu (\partial_\mu \partial_\beta v^\beta) + d_3^{(11)} H^{\mu\nu} v^\rho (\partial_\mu \partial_\rho v_\nu) \\
&\quad + d_4^{(11)} \epsilon_{\mu\alpha\rho\beta} H^{\mu\nu} v^\alpha (\partial^\rho \partial_\nu v^\beta) \\
U^{(12)} &= d_1^{(12)} \Phi (\partial_\alpha v^\alpha)^2 + d_2^{(12)} \Phi (\partial^\mu v^\alpha) (\partial_\alpha v_\mu) + d_3^{(12)} \epsilon_{\mu\nu\alpha\beta} \Phi (\partial^\mu v^\alpha) (\partial^\nu v^\beta) \\
U^{(13)} &= d^{(13)} \Phi v^\alpha (\partial_\alpha \partial_\beta v^\beta) \\
U^{(14)} &= d_1^{(14)} v^\alpha (\partial^\beta v_\beta) (\partial_\alpha \partial_\nu v^\nu) + d_2^{(14)} v^\alpha (\partial_\alpha v_\beta) (\partial^\beta \partial^\nu v_\nu) \\
&\quad + d_3^{(14)} \epsilon_{\mu\nu\alpha\beta} v^\alpha (\partial^\mu v_\beta) (\partial^\nu \partial^\rho v_\rho) + d_4^{(14)} \epsilon_{\mu\nu\alpha\beta} v_\rho (\partial^\mu v^\beta) (\partial^\nu \partial^\rho v^\alpha) \\
&\quad + d_5^{(14)} \epsilon_{\mu\nu\alpha\beta} v^\alpha (\partial^\mu v^\rho) (\partial^\nu \partial_\rho v^\beta) \\
U^{(15)} &= d^{(15)} v^\alpha v^\beta (\partial_\alpha \partial_\beta \partial_\mu v^\mu). \tag{3.7}
\end{aligned}$$

We have discarded a lot of terms because up to a total derivatives they are of the type already considered. As a general strategy, we have eliminated all terms with derivatives on the ghost fields  $u_\mu$ . It is somewhat more complicated to prove that one can make  $c_3^{(6)} = 0$  if one subtracts a total divergence and redefines  $c_4^{(6)}, c_5^{(6)}, c_6^{(6)}$ , and  $c_9^{(6)} = 0$  if one subtract a total divergence and redefines  $c_7^{(6)}, c_8^{(6)}$ .

Also because

$$\begin{aligned}
(\partial^2 + m_j^2) f_j &= 0, \quad j = 1, 2, 3 \implies \\
(\partial^\mu f_1) (\partial_\mu f_2) f_3 &= \frac{1}{2} (m_1^2 + m_2^2 - m_3^2) f_1 f_2 f_3 + \frac{1}{2} \partial_\mu \left[ (\partial^\mu f_1) f_2 f_3 + f_1 (\partial^\mu f_2) f_3 - f_1 f_2 (\partial^\mu f_3) \right] \tag{3.8}
\end{aligned}$$

we can eliminate many terms by subtracting a total divergence and redefining other terms of lower canonical dimension.

Now we can put to zero some of the constants above if we subtract from  $T$  a coboundary i.e. an expression of the form  $d_Q B$  where we take  $B$  to be a Wick polynomial with the following properties:

$$\begin{aligned}
U_g B &= B U_g, \quad \forall g \in \mathcal{P} \\
gh(T) &= -1 \\
2 &\leq \deg(T) \leq 4. \tag{3.9}
\end{aligned}$$

We have the following admissible expressions:

First we have terms which do also appear for the massless case i.e. without the ghost field  $v_\mu$  namely:

$$\begin{aligned}
B^{(1)} &= b_1^{(1)} H^{\mu\nu} (\partial_\rho H_{\mu\nu}) \tilde{u}^\rho + b_2^{(1)} H^{\mu\nu} (\partial_\mu H_{\mu\rho}) \tilde{u}^\rho \\
&+ b_3^{(1)} H^{\mu\nu} (\partial^\rho H_{\mu\rho}) \tilde{u}_\nu + b_4^{(1)} \epsilon_{\mu\rho\alpha\beta} H^{\mu\nu} (\partial^\alpha H^\rho{}_{\cdot\nu}) \tilde{u}^\beta
\end{aligned}$$

$$\begin{aligned}
B^{(2)} &= b^{(2)} \Phi (\partial_\mu H^{\mu\nu}) \tilde{u}_\nu \\
B^{(3)} &= b^{(3)} \Phi H^{\mu\nu} (\partial_\mu \tilde{u}_\nu) \\
B^{(4)} &= b^{(4)} \Phi^2 \partial^\mu \tilde{u}_\mu \\
B^{(5)} &= b_1^{(5)} u^\mu \tilde{u}_\mu (\partial_\rho \tilde{u}^\rho) + b_2^{(5)} u^\mu \tilde{u}^\nu (\partial_\mu \tilde{u}_\nu) + b_3^{(5)} u^\mu \tilde{u}^\nu (\partial_\nu \tilde{u}_\mu) + b_4^{(5)} \epsilon_{\mu\nu\rho\sigma} u^\mu \tilde{u}^\nu (\partial^\rho \tilde{u}^\sigma); \quad (3.10)
\end{aligned}$$

then we have the terms containing at least one factor  $v_\mu$  namely:

$$\begin{aligned}
V^{(1)} &= f^{(1)} H^{\mu\nu} v_\mu \tilde{u}_\nu \\
V^{(2)} &= f^{(2)} \Phi v^\mu \tilde{u}_\mu \\
V^{(3)} &= f_1^{(3)} v^\alpha (\partial_\mu v_\alpha) \tilde{u}^\mu + f_2^{(3)} v_\mu (\partial^\alpha v_\alpha) \tilde{u}^\mu + f_3^{(3)} v_\nu (\partial^\nu v^\mu) \tilde{u}_\mu + f_4^{(3)} \epsilon_{\mu\nu\alpha\beta} v^\alpha (\partial^\nu v^\beta) \tilde{u}^\mu. \quad (3.11)
\end{aligned}$$

We have discarded some terms because up to a total derivatives they are of the type already considered. Now one can prove that:

- One can use  $B^{(1)}$  to make  $c_2^{(6)}, c_4^{(6)}, c_5^{(6)}, c_7^{(6)}$  equal to zero. In this way one redefines  $T^{(7)}, T^{(8)}, T^{(11)}, T^{(12)}, T^{(13)}$ .
- One can use  $B^{(2)}$  to make  $c_1^{(7)}$  equal to zero. In this way one redefines  $T^{(9)}, T^{(11)}, T^{(12)}, T^{(14)}, T^{(15)}, T^{(16)}$ .
- One can use  $B^{(3)}$  to make  $c_2^{(13)}$  equal to zero. In this way one redefines  $T^{(8)} - T^{(11)}, T^{(14)}, T^{(15)}$ .
- One can use  $B^{(4)}$  to make  $c^{(15)}$  equal to zero. In this way one redefines  $T^{(9)}, T^{(14)}$ .
- One can use  $B^{(5)}$  to make  $c_1^{(12)}, c_2^{(12)}, c_3^{(12)}, c_4^{(12)}$  equal to zero. In this way one redefines  $T^{(11)}, T^{(14)}, T^{(16)}$ .
- One can use  $V^{(1)}$  to make  $d_2^{(3)}$  equal to zero. In this way one redefines  $T^{(4)}, U^{(1)}, U^{(4)}, U^{(7)}, U^{(8)}$ .
- One can use  $V^{(2)}$  to make  $d^{(5)}$  equal to zero. In this way one redefines  $T^{(5)}, U^{(2)}, U^{(6)}, U^{(7)}, U^{(8)}$ .
- One can use  $V^{(3)}$  to make XX  $d_1^{(7)}, d_2^{(7)}, d_3^{(7)}, d_4^{(7)}$  equal to zero. In this way one redefines  $U^{(7)} - U^{(10)}, U^{(12)}, U^{(13)}$ .

One can count that we are left with 37 coefficients of type  $c^{(j)}$  and 23 coefficients of type  $d^{(j)}$ . So we have 60 free parameters which should be fixed by the condition of gauge invariance. Now one considers the coupling

$$T = \sum_{j=1}^{16} T^{(j)} + \sum_{j=1}^{13} U^{(j)} \quad (3.12)$$

and computes the expression  $d_Q T$ . It is convenient to follow the same strategy as above and eliminate, up to a divergence, the derivatives on the ghost fields  $u_\mu$ . One gets by a straightforward but tedious computation the following result:

$$d_Q T = i \partial_\mu X^\mu + i u^\mu Y_\mu + i Z_4 + i Z_6 \quad (3.13)$$

where the expressions  $X^\mu, Y^\mu$  do not contain ghost fields and the expressions  $Z_j$  are tri-linear in the ghost fields of canonical dimension  $j = 4, 6$ . The explicit expression of  $X^\mu$  and  $Z_j$  are not important for the moment. The expression of  $Y^\mu$  is rather long: we group the various sectors:

$$Y_\mu = \sum_{j=1}^{22} Y_\mu^j \quad (3.14)$$

where

$$Y_\mu^1 \equiv \left[ 3c^{(1)} + m^2 c_6^{(6)} - \frac{1}{2} m d_3^{(3)} \right] H^{\nu\rho} (\partial_\nu H_{\mu\rho}) + \left[ 3c^{(1)} - c^{(4)} + m^2 c_6^{(6)} \right] H_{\mu\nu} (\partial_\rho H^{\nu\rho}) \\ - \left[ \frac{3}{2} c^{(1)} + c^{(2)} + m^2 c_1^{(6)} + \frac{1}{2} m d_1^{(3)} \right] H^{\nu\rho} (\partial_\mu H_{\nu\rho}) + \frac{1}{2} m d_4^{(3)} \epsilon_{\mu\nu\rho\sigma} H^{\sigma\lambda} (\partial^\rho H_{\lambda}^\nu) \quad (3.15)$$

$$Y_\mu^2 \equiv \left[ 2c^{(2)} - c^{(5)} + m^2 c_2^{(7)} - m^2 c^{(8)} + \frac{1}{2} m^2 c_1^{(11)} \right] \Phi (\partial^\nu H_{\mu\nu}) \quad (3.16)$$

$$Y_\mu^3 \equiv \left[ 2c^{(2)} - \frac{1}{2} c^{(4)} + m^2 c_2^{(7)} - \frac{1}{2} m^2 c^{(8)} + \frac{1}{2} m^2 c_1^{(13)} - \frac{1}{2} m d^{(4)} \right] H^{\mu\nu} (\partial_\nu \Phi) \quad (3.17)$$

$$Y_\mu^4 \equiv - \left[ 3c^{(3)} + \frac{1}{2} c^{(5)} + \frac{3}{2} m^2 c^{(9)} + m^2 c^{(10)} - \frac{1}{2} m^2 c_1^{(14)} + \frac{1}{2} m d^{(6)} \right] \Phi (\partial_\mu \Phi) \quad (3.18)$$

$$Y_\mu^5 \equiv \frac{1}{2} c_1^{(6)} (\partial_\mu \partial_\nu H_{\rho\sigma}) (\partial^\nu H^{\rho\sigma}) + \left[ -2c_1^{(6)} + c_6^{(6)} - c_4^{(11)} \right] (\partial_\nu \partial^\rho H^{\nu\sigma}) (\partial_\sigma H_{\mu\rho}) \\ - \left[ 2c_1^{(6)} + c_2^{(11)} \right] (\partial_\sigma \partial^\rho H_{\mu\rho}) (\partial_\nu H^{\nu\sigma}) + \left[ -2c_1^{(6)} + c_6^{(6)} \right] (\partial_\sigma \partial_\rho H_{\mu\rho}) (\partial^\rho H^{\nu\sigma}) \\ - \left[ \frac{1}{2} c_6^{(6)} + c_2^{(7)} \right] (\partial_\mu \partial_\rho H_{\nu\sigma}) (\partial^\sigma H^{\nu\rho}) - c_6^{(6)} (\partial_\rho \partial_\sigma H_{\mu\nu}) (\partial^\sigma H^{\nu\rho}) - c_6^{(6)} (\partial_\rho \partial_\sigma H^{\nu\rho}) (\partial^\sigma H_{\mu\nu}) \\ - \left[ \frac{1}{2} c_6^{(6)} + \frac{1}{2} c^{(8)} + c_5^{(11)} \right] (\partial^\nu \partial_\nu H^{\nu\sigma}) (\partial_\mu H^{\rho\sigma}) - \left[ c_6^{(6)} + c_3^{(11)} \right] (\partial_\mu \partial^\rho H_{\rho\sigma}) (\partial_\nu H^{\nu\sigma}) \\ + 2c_8^{(6)} \epsilon_{\nu\rho\sigma\lambda} (\partial_\nu \partial_\tau H_{\mu}^\sigma) (\partial^\lambda H^{\rho\tau}) + 2c_8^{(6)} \epsilon_{\mu\rho\sigma\lambda} (\partial_\nu \partial^\lambda H^{\rho\tau}) (\partial_\tau H^{\nu\sigma}) \\ + \left[ 2c_8^{(6)} + c_7^{(11)} \right] \epsilon_{\mu\alpha\beta\rho} (\partial_\nu \partial_\lambda H^{\lambda\beta}) (\partial^\rho H^{\nu\alpha}) + \left[ 2c_8^{(6)} - c_3^{(7)} \right] \epsilon_{\alpha\beta\rho\lambda} (\partial_\mu \partial^\lambda H^{\nu\beta}) (\partial^\rho H_{\nu}^\alpha) \\ + \left[ c_8^{(6)} - c_3^{(11)} \right] \epsilon_{\mu\rho\alpha\lambda} (\partial_\sigma \partial^\lambda H^{\rho\sigma}) (\partial_\nu H^{\alpha\nu}) - c_1^{(11)} (\partial^\rho \partial^\sigma H_{\rho\sigma}) (\partial^\nu H_{\mu\nu}) \\ - c_6^{(11)} \epsilon_{\mu\sigma\alpha\rho} (\partial^\sigma \partial^\lambda H_{\nu\lambda}) (\partial^\rho H^{\alpha\nu}) \quad (3.19)$$

$$Y_\mu^6 \equiv - \left[ 2c_1^{(6)} + c_3^{(13)} \right] H^{\nu\sigma} (\partial_\nu \partial_\sigma \partial^\rho H_{\mu\rho}) - \frac{1}{2} \left[ c_1^{(6)} + c^{(8)} \right] H^{\nu\sigma} (\partial_\mu \partial_\nu \partial^\rho H_{\rho\sigma}) \\ + \left[ c_8^{(6)} - c_4^{(13)} \right] \epsilon_{\mu\rho\alpha\lambda} H^{\nu\alpha} (\partial_\sigma \partial_\nu \partial^\lambda H^{\rho\sigma}) - c_1^{(13)} H_{\mu\nu} (\partial^\nu \partial^\rho \partial^\sigma H_{\rho\sigma}) \quad (3.20)$$

$$Y_\mu^7 \equiv - \left[ c_2^{(7)} + \frac{1}{2} c_4^{(11)} \right] (\partial^\rho \partial^\sigma \Phi) (\partial_\sigma H_{\mu\rho}) + \left[ -c_2^{(7)} + \frac{1}{4} c^{(8)} + c^{(10)} - \frac{1}{2} c_5^{(11)} \right] (\partial^\sigma \partial^\rho \Phi) (\partial_\mu H_{\rho\sigma}) \\ + \left[ \frac{1}{2} c_2^{(7)} + \frac{1}{2} c^{(8)} + c^{(10)} - \frac{1}{2} c_2^{(11)} - \frac{1}{2} c_3^{(11)} - c^{(16)} \right] (\partial_\mu \partial_\nu \Phi) (\partial_\rho H^{\nu\rho}) \\ + \left[ c_3^{(7)} - \frac{1}{2} c_6^{(11)} - \frac{1}{2} c_7^{(11)} \right] \epsilon_{\mu\nu\rho\beta} (\partial^\nu \partial_\sigma \Phi) (\partial^\beta H^{\rho\sigma}) \quad (3.21)$$

$$Y_\mu^8 \equiv \left[ -c_2^{(7)} + \frac{1}{2} c^{(8)} \right] (\partial_\sigma \Phi) (\partial^\rho \partial^\sigma H_{\mu\rho}) + \left[ -\frac{3}{2} c_2^{(7)} + c^{(8)} + c^{(10)} - c_2^{(14)} \right] (\partial_\rho \Phi) (\partial_\sigma \partial_\mu H^{\rho\sigma}) \\ + \left[ \frac{1}{2} c_2^{(7)} - c^{(9)} - c_1^{(14)} \right] (\partial_\mu \Phi) (\partial_\nu \partial_\rho H^{\nu\rho}) + \left[ c_3^{(7)} + c_4^{(13)} \right] \epsilon_{\mu\nu\rho\beta} (\partial^\nu \Phi) (\partial_\sigma \partial^\beta H^{\rho\sigma}) \quad (3.22)$$

$$Y_\mu^9 \equiv \left[ -\frac{1}{2} c_2^{(7)} + \frac{1}{2} c^{(8)} - c^{(9)} \right] \Phi (\partial_\mu \partial_\nu \partial_\rho H^{\nu\rho}) \quad (3.23)$$

$$Y_\mu^{10} \equiv \left[ \frac{1}{4} c^{(8)} + c^{(10)} - \frac{1}{2} c_3^{(13)} \right] H^{\rho\sigma} (\partial_\mu \partial_\rho \partial_\sigma \Phi) \quad (3.24)$$

$$Y_\mu^{11} \equiv \frac{1}{2} \left[ c^{(10)} - c_2^{(14)} - c^{(16)} \right] (\partial^\nu \Phi) (\partial_\mu \partial_\nu \Phi) \quad (3.25)$$

$$Y_\mu^{12} \equiv - \left[ mc^{(4)} + md^{(1)} + \frac{1}{2} m^2 d_3^{(3)} \right] H_{\mu\nu} v^\nu \quad (3.26)$$

$$Y_\mu^{13} \equiv - \left[ mc^{(5)} + md^{(2)} + \frac{1}{2} m^2 d^{(4)} - \frac{1}{2} m^2 d_2^{(8)} \right] \Phi v_\mu \quad (3.27)$$

$$Y_\mu^{14} \equiv - \left[ mc_1^{(11)} + d_1^{(3)} + \frac{1}{2} d_3^{(3)} + d_1^{(7)} - \frac{1}{2} md_2^{(10)} \right] (\partial^\nu H_{\mu\nu}) (\partial_\alpha v^\alpha) \\ - \left[ mc_2^{(11)} + d_3^{(3)} + d_3^{(7)} - md_1^{(10)} \right] (\partial_\rho H^{\nu\rho}) (\partial_\nu v_\mu) \\ - \left[ mc_3^{(11)} - \frac{1}{2} d^{(4)} + d_2^{(7)} - \frac{1}{2} md_3^{(10)} \right] (\partial_\rho H^{\nu\rho}) (\partial_\mu v_\nu) \\ - \left[ mc_4^{(11)} + \frac{1}{2} d_3^{(3)} - \frac{1}{2} md_3^{(10)} \right] (\partial_\rho H_{\mu\nu}) (\partial^\nu v^\rho) \\ - \left[ mc_5^{(11)} - \frac{3}{4} d_3^{(3)} - \frac{1}{2} d^{(4)} - \frac{1}{2} md_2^{(10)} \right] (\partial_\mu H_{\rho\sigma}) (\partial^\rho v^\sigma) \\ + \frac{1}{2} d_3^{(3)} (\partial_\nu H_{\mu\rho}) (\partial^\nu A_\rho) + \left[ c_6^{(11)} - \frac{1}{2} md_4^{(10)} \right] \epsilon_{\nu\rho\alpha\beta} (\partial^\rho H^{\alpha\nu}) (\partial^\beta v_\nu)$$

$$\begin{aligned}
& + \left[ c_7^{(11)} - \frac{1}{2}d_4^{(3)} - \frac{1}{2}md_5^{(10)} \right] \epsilon_{\mu\rho\alpha\beta}(\partial^\rho H^{\alpha\nu})(\partial_\nu v^\beta) \\
& - \left[ c_8^{(11)} - \frac{1}{2}d_4^{(3)} + d_4^{(7)} - \frac{1}{2}md_5^{(10)} \right] \epsilon_{\mu\alpha\sigma\beta}(\partial_\nu H^{\alpha\nu})(\partial^\sigma v^\beta) \\
& - \frac{1}{2} \left[ 3d_4^{(3)} + md_4^{(10)} \right] \epsilon_{\alpha\rho\nu\beta}(\partial^\alpha H_\mu^\rho)(\partial^\nu v^\beta)
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
Y_\mu^{15} \equiv & - \left[ mc_1^{(13)} + d_1^{(3)} + \frac{1}{2}d_3^{(3)} - \frac{1}{2}md_2^{(10)} - \frac{1}{2}md_3^{(10)} \right] H_{\mu\nu}(\partial^\nu \partial^\alpha v_\alpha) \\
& - \left[ mc_3^{(13)} + \frac{1}{2}d_3^{(3)} - md_1^{(10)} \right] H^{\rho\sigma}(\partial_\rho \partial_\sigma v_\mu) \\
& + \frac{1}{2} \left[ \frac{1}{2}d_3^{(3)} + d^{(4)} + md_2^{(10)} + md_3^{(10)} \right] H^{\nu\rho}(\partial_\mu \partial_\nu v_\rho) \\
& - \left[ mc_4^{(13)} - \frac{1}{2}md_4^{(3)} - md_5^{(10)} \right] \epsilon_{\mu\alpha\rho\beta} H^{\alpha\nu}(\partial_\nu \partial^\rho v^\beta)
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
Y_\mu^{16} \equiv & - \left[ mc_1^{(14)} - \frac{1}{2}d^{(4)} - \frac{1}{2}d^{(6)} + \frac{1}{2}d_1^{(7)} - md_1^{(12)} + \frac{1}{2}md^{(13)} \right] (\partial_\mu \Phi)(\partial^\alpha v_\alpha) \\
& - \left[ mc_2^{(14)} + \frac{1}{4}d^{(4)} + \frac{1}{2}d_2^{(7)} - md_2^{(12)} + \frac{1}{2}d^{(13)} \right] (\partial^\alpha \Phi)(\partial_\mu v_\alpha) \\
& + \frac{1}{2} \left[ d^{(4)} - d_3^{(7)} \right] (\partial^\nu \Phi)(\partial_\nu v_\mu) \\
& + m \left[ c_3^{(14)} - \frac{1}{2}d_4^{(7)} + d_3^{(12)} \right] \epsilon_{\mu\nu\alpha\beta}(\partial^\nu \Phi)(\partial^\alpha v^\beta)
\end{aligned} \tag{3.30}$$

$$Y_\mu^{17} \equiv - \left[ mc^{(16)} - \frac{1}{4}d^{(4)} + \frac{1}{2}d_1^{(8)} + \frac{1}{2}d_3^{(8)} + \frac{1}{2}md^{(13)} \right] (\partial_\mu \partial_\nu \Phi)v^\nu \tag{3.31}$$

$$\begin{aligned}
Y_\mu^{18} \equiv & \left[ d^{(1)} - md_1^{(7)} - md_2^{(8)} + \frac{1}{2}md^{(9)} - \frac{1}{2}m^2d_2^{(10)} \right] v_\mu(\partial^\alpha v_\alpha) \\
& + \left[ d^{(1)} - md_3^{(7)} - md_1^{(8)} - \frac{1}{2}m^2d_3^{(10)} \right] v^\alpha(\partial_\alpha v_\mu) \\
& - \left[ \frac{1}{2}d^{(1)} + d^{(2)} + md_2^{(7)} + md_3^{(8)} + \frac{1}{2}md^{(9)} + m^2d_1^{(10)} \right] v^\alpha(\partial_\mu v_\alpha) \\
& - m \left[ d_4^{(7)} + d_4^{(8)} + \frac{1}{2}md_5^{(10)} \right] \epsilon_{\mu\nu\alpha\beta}v^\nu(\partial^\alpha v^\beta)
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
Y_\mu^{19} \equiv & \left[ \frac{1}{2}d_3^{(3)} + \frac{1}{2}d^{(4)} - d_3^{(8)} \right] v^\rho(\partial_\mu \partial_\nu H_{\nu\rho}) - \left[ \frac{1}{2}d_3^{(3)} + d_1^{(8)} \right] v_\rho(\partial^\nu \partial^\rho H_{\mu\nu}) \\
& - \left[ \frac{1}{2}d_3^{(3)} + d_2^{(8)} \right] v_\mu(\partial_\mu \partial_\rho H^{\nu\rho}) - \left[ d_4^{(3)} + d_4^{(8)} \right] \epsilon_{\mu\alpha\rho\beta}v^\beta(\partial^\alpha \partial_\nu H^{\nu\rho})
\end{aligned} \tag{3.33}$$

$$Y_\mu^{20} \equiv \left[ \frac{1}{2} d^{(6)} + m d_1^{(12)} + m d_2^{(12)} - m d^{(13)} \right] \Phi(\partial_\mu \partial_\nu v^\nu) \quad (3.34)$$

$$\begin{aligned} Y_\mu^{21} \equiv & \frac{1}{2} d_1^{(10)} (\partial^\nu v^\alpha) (\partial_\mu \partial_\nu v_\alpha) + \frac{1}{2} \left[ d_2^{(10)} + d_3^{(10)} - d^{(13)} \right] (\partial_\mu v_\nu) (\partial^\nu \partial^\rho v_\rho) \\ & + \frac{1}{2} d_2^{(10)} (\partial_\nu v_\mu) (\partial^\nu \partial^\rho v_\rho) + \frac{1}{2} d_3^{(10)} (\partial^\nu v^\rho) (\partial_\nu \partial_\rho v_\mu) \\ & - d_1^{(12)} (\partial_\nu v^\nu) (\partial_\mu \partial_\rho v^\rho) - d_2^{(12)} (\partial^\beta v^\alpha) (\partial_\mu \partial_\alpha v_\beta) \\ & + \frac{1}{2} d_4^{(10)} \epsilon_{\mu\rho\alpha\beta} (\partial^\alpha v^\beta) (\partial^\rho \partial^\nu v_\nu) + \frac{1}{2} d_4^{(10)} \epsilon_{\mu\rho\alpha\beta} (\partial^\rho v^\nu) (\partial_\nu \partial^\alpha v^\beta) \\ & + \left[ \frac{1}{2} d_4^{(10)} + d_3^{(12)} \right] \epsilon_{\rho\nu\alpha\beta} (\partial^\alpha v^\beta) (\partial_\mu \partial^\rho v^\nu) + \frac{1}{2} d_5^{(10)} \epsilon_{\mu\rho\alpha\beta} (\partial_\nu v^\beta) (\partial^\nu \partial^\rho v^\alpha) \end{aligned} \quad (3.35)$$

$$Y_\mu^{22} \equiv -\frac{1}{2} d^{(13)} v^\alpha (\partial_\mu \partial_\alpha \partial_\beta v^\beta). \quad (3.36)$$

We now impose the gauge invariance condition (2.46). It is sufficient to take  $T^\mu$  a tri-linear expression in the fields  $H_{\mu\nu}, \Phi, u_\mu, \tilde{u}_\mu, A_\mu$  verifying the following conditions:

$$\begin{aligned} U_g T^\mu U_g^{-1} &= \Lambda^{\mu\nu} T_\nu, \quad \forall g \in \mathcal{P} \\ gh(T) &= 1 \\ 3 &\leq \deg(T^\mu) \leq 5. \end{aligned} \quad (3.37)$$

If we define

$$\tilde{T}^\mu \equiv T^\mu - X^\mu \quad (3.38)$$

then we get from (2.46) and (3.13):

$$u^\mu Y_\mu + Z_4 + Z_6 = \partial_\mu \tilde{T}^\mu. \quad (3.39)$$

The generic form for  $\tilde{T}^\mu$  is

$$\tilde{T}^\mu = u_\nu T^{\mu\nu} + (\partial_\rho u_\nu) T^{\mu\nu\rho} + (\partial_\rho \partial_\sigma u_\nu) T^{\mu\nu\rho\sigma} + d_0 u^\mu u^\nu \tilde{u}_\nu + d_1 \epsilon^{\mu\nu\rho\sigma} u_\nu u_\rho \tilde{u}_\sigma + S^\mu \quad (3.40)$$

where the expressions  $T^{\mu\nu}, T^{\mu\nu\rho}, T^{\mu\nu\rho\sigma}$  are bi-linear in the fields  $H_{\mu\nu}, \Phi, A_\mu$ , the expression  $S^\mu$  is tri-linear in the ghost fields and of canonical dimension 5 and  $d_0, d_1$  are constants. We have by direct computation:

$$\begin{aligned} \partial_\mu \tilde{T}^\mu &= u_\nu (\partial_\mu T^{\mu\nu}) + (\partial_\rho u_\nu) (T^{\rho\nu} + \partial_\mu T^{\mu\nu\rho}) + (\partial_\rho \partial_\sigma u_\nu) (\partial_\mu T^{\mu\nu\rho\sigma} + T^{\sigma\nu\rho}) \\ &+ (\partial_\mu \partial_\rho \partial_\sigma u_\nu) T^{\mu\nu\rho\sigma} + d_0 [(\partial_\mu u^\mu) u^\nu \tilde{u}_\nu + u^\mu (\partial_\mu u^\nu) \tilde{u}_\nu + u^\mu u^\nu (\partial_\mu \tilde{u}_\nu)] \\ &+ d_1 \epsilon^{\mu\nu\rho\sigma} [2(\partial_\mu u_\nu) u_\rho \tilde{u}_\sigma + u_\nu u_\rho (\partial_\mu \tilde{u}_\sigma)] + \partial_\mu S^\mu. \end{aligned} \quad (3.41)$$

Let us write

$$T^{\mu\nu\rho\sigma} = T_1^{\mu\nu\rho\sigma} + T_2^{\mu\nu\rho\sigma} \quad (3.42)$$



where the tensor  $T_1^{\mu\nu\rho\sigma}$  does not contain terms with the factors  $\eta^{\mu\rho}, \eta^{\mu\sigma}, \eta^{\rho\sigma}$  and  $T_2^{\mu\nu\rho\sigma}$  collects all terms containing at least one of these factors. The terms with the factor  $\eta^{\mu\rho}$  can be discarded if we redefine  $T^{\mu\nu}$ . It follows that the generic form is

$$T_2^{\mu\nu\rho\sigma} = \frac{1}{2}(\eta^{\mu\rho}T_2^{\nu\sigma} + \eta^{\mu\sigma}T_2^{\nu\rho}). \quad (3.43)$$

We also write

$$T^{\mu\nu\rho} = T_1^{\mu\nu\rho} + \eta^{\mu\rho}T^\nu \quad (3.44)$$

where  $T_1^{\mu\nu\rho}$  collects all terms without the factor  $\eta^{\mu\rho}$ .

Then we obtain from (3.41)

$$\begin{aligned} \partial_\mu \tilde{T}^\mu = & u_\nu (\partial_\mu T^{\mu\nu} - m^2 T^\nu) + (\partial_\rho u_\nu) (T^{\rho\nu} + \partial_\mu T_1^{\mu\nu\rho} + \partial^\rho T^\nu - m^2 T_2^{\rho\nu}) \\ & + (\partial_\rho \partial_\sigma u_\nu) (\partial_\mu T_1^{\mu\nu\rho\sigma} + T_1^{\sigma\nu\rho} + \partial^\rho T_2^{\sigma\nu}) + (\partial_\mu \partial_\rho \partial_\sigma u_\nu) T_1^{\mu\nu\rho\sigma} \\ & + d_0 [(\partial_\mu u^\mu) u^\nu \tilde{u}_\nu + u^\mu (\partial_\mu u_\nu) \tilde{u}^\nu + u^\mu u^\nu (\partial_\mu \tilde{u}_\nu)] \\ & + d_1 \epsilon^{\mu\nu\rho\sigma} [2(\partial_\mu u_\nu) u_\rho \tilde{u}_\sigma + u_\nu u_\rho (\partial_\mu \tilde{u}_\sigma)] + \partial_\mu S^\mu. \end{aligned} \quad (3.45)$$

It follows that the equation (3.39) is equivalent to the following system:

$$\begin{aligned} \mathcal{S}_{\mu\rho\sigma}(T_1^{\mu\nu\rho\sigma}) &= 0 \\ T_{1+}^{\mu\nu\rho} &= -\partial_\sigma T_1^{\sigma\nu\rho\mu} - \frac{1}{2}(\partial^\rho T_2^{\mu\nu} + \partial^\mu T_2^{\rho\nu}) \\ T^{\rho\nu} &= -\partial_\mu T_1^{\mu\nu\rho} - \partial^\rho T^\nu + m^2 T_2^{\rho\nu} \\ Y^\nu &= \partial_\mu T^{\mu\nu} - m^2 T^\nu \\ d_Q [(\partial_\mu u^\mu) u^\nu \tilde{u}_\nu + u^\mu (\partial_\mu u_\nu) \tilde{u}^\nu + u^\mu u^\nu (\partial_\mu \tilde{u}_\nu)] \\ &+ d_1 \epsilon^{\mu\nu\rho\sigma} [2(\partial_\mu u_\nu) u_\rho \tilde{u}_\sigma + u_\nu u_\rho (\partial_\mu \tilde{u}_\sigma)] + (\partial_\mu S^\mu)_4 = Z_4 \\ &(\partial_\mu S^\mu)_6 = Z_6. \end{aligned} \quad (3.46)$$

Here we have written

$$T_1^{\mu\nu\rho} = T_{1+}^{\mu\nu\rho} + T_{1-}^{\mu\nu\rho} \quad (3.47)$$

where the two pieces have the following symmetry properties:

$$T_{1\epsilon}^{\mu\nu\rho} = \epsilon T_{1\epsilon}^{\rho\nu\mu} \quad \forall \epsilon = \pm. \quad (3.48)$$

The expression  $(\partial_\mu S^\mu)_j$  contains the terms of canonical dimension  $j = 4, 6$  of  $\partial_\mu S^\mu$ ; the expression  $(\partial_\mu S^\mu)_4$  is proportional to the mass, so is zero in the massless case.

From the first four equations of the system (3.46) we obtain

$$Y^\nu = (\partial^2 + m^2)(\partial_\mu T_1^{\mu\nu} - T^\nu), \quad (3.49)$$

so  $Y_\mu$  has the generic form

$$Y_\mu = \frac{1}{2}(\partial^2 + m^2)Z_\mu \quad (3.50)$$

where  $Z_\mu$  is a Wick polynomial bilinear in the fields  $H_{\mu\nu}, \Phi, A_\mu$  and verifies the following conditions:

$$\begin{aligned} U_g Z^\mu U_g^{-1} &= \Lambda^{\mu\nu} Z_\nu, \quad \forall g \in \mathcal{P} \\ gh(T) &= 0 \\ 2 \leq \deg(T^\mu) &\leq 3. \end{aligned} \quad (3.51)$$

The generic form of  $Z_\mu$  is

$$\begin{aligned} Z_\mu &= f_1 \Phi(\partial_\mu \Phi) + f_2 H^{\alpha\beta}(\partial_\mu H_{\alpha\beta}) + f_3 \Phi(\partial^\nu H_{\mu\nu}) + f_4 H_{\mu\nu}(\partial^\nu \Phi) + f_5 H_{\mu\nu}(\partial_\rho H^{\nu\rho}) \\ &\quad + f_6 H^{\nu\rho}(\partial_\rho H_{\mu\nu}) + f_7 \epsilon_{\mu\nu\alpha\beta} H^{\alpha\rho}(\partial^\nu H_\rho^\beta) + f_8 \Phi v_\mu + f_9 H_{\mu\nu} v^\nu + f_{10} v_\mu(\partial_\alpha v_\alpha) \\ &\quad + f_{11} v^\alpha(\partial_\mu v_\alpha) + f_{12} v^\alpha(\partial_\alpha v^\mu) + f_{13} \epsilon_{\mu\nu\alpha\beta} v^\alpha(\partial^\nu v^\beta). \end{aligned} \quad (3.52)$$

The basic equation (3.50) becomes now

$$\begin{aligned} Y_\mu &= f_1(\partial^\nu \Phi)(\partial_\mu \partial_\nu \Phi) + f_2(\partial^\nu H^{\alpha\beta})(\partial_\mu \partial_\nu H_{\alpha\beta}) + f_3(\partial_\rho \Phi)(\partial^\nu \partial^\rho H_{\mu\nu}) + f_4(\partial_\rho H_{\mu\nu})(\partial^\nu \partial^\rho \Phi) \\ &\quad + f_5(\partial^\sigma H_{\mu\nu})(\partial_\rho \partial_\sigma H^{\nu\rho}) + f_6(\partial^\sigma H^{\nu\rho})(\partial_\rho \partial_\sigma H_{\mu\nu}) + f_7 \epsilon_{\mu\nu\alpha\beta}(\partial^\sigma H^{\alpha\rho})(\partial^\nu \partial_\sigma H_\rho^\beta) \\ &\quad + f_8(\partial^\nu \Phi)(\partial_\nu v_\mu) + f_9(\partial_\rho H_{\mu\nu})(\partial^\rho v^\nu) + f_{10}(\partial_\nu v_\mu)(\partial^\nu \partial_\alpha v_\alpha) \\ &\quad + f_{11}(\partial^\nu v^\alpha)(\partial_\mu \partial_\nu v_\alpha) + f_{12}(\partial^\nu v^\alpha)(\partial_\nu \partial_\alpha v^\mu) + f_{13} \epsilon_{\mu\nu\alpha\beta}(\partial^\rho v^\alpha)(\partial_\rho \partial^\nu v^\beta) - \frac{1}{2} m^2 Z_\mu. \end{aligned} \quad (3.53)$$

If we substitute here the expressions (3.14) - (3.36) we get the a system of equations of 78 equations for 73 the unknowns  $c^{(j)}, d^{(j)}, f_j$  ( $37 + 23 + 13 = 73$ ). One can solve this system explicitly, the only non-zero coefficients are:

$$\begin{aligned} c^{(1)} &= \frac{2}{3} m^2 a - \frac{1}{3} m^2 b \quad c^{(2)} = \frac{1}{2} m^2 a \quad c^{(3)} = -\frac{3}{4} m^2 a \quad c^{(4)} = -m^2 b \quad c^{(5)} = -m^2 a. \\ c_1^{(6)} &= -2a \quad c_6^{(6)} = -4a \quad c_2^{(7)} = 2a \quad c^{(8)} = 4a \quad c^{(9)} = a \\ c^{(10)} &= a \quad c_2^{(11)} = 4a \quad c_3^{(11)} = 4a \quad c_3^{(13)} = 4a \quad c_2^{(14)} = 2a \\ d^{(1)} &= \frac{3}{2} m^2 b \quad d^{(2)} = m^2 a \quad d_1^{(3)} = mb \quad d_3^{(3)} = -2mb \quad d^{(4)} = mb \\ d_1^{(7)} &= mb \quad d_2^{(7)} = -4ma - \frac{1}{2} mb \quad d_3^{(7)} = mb \quad d_1^{(8)} = mb \quad d_2^{(8)} = mb \quad d_3^{(8)} = -\frac{1}{2} mb \\ d^{(9)} &= 2mb \quad d_1^{(10)} = 4a - b \quad d_2^{(10)} = 2b \quad d_3^{(10)} = -2b \end{aligned} \quad (3.54)$$

and

$$\begin{aligned} f_1 &= -\frac{1}{2} a \quad f_2 = -a \quad f_4 = -2a \quad f_5 = 4a \quad f_6 = 4a; \\ f_9 &= -mb \quad f_{10} = b \quad f_{11} = 2a - \frac{1}{2} b \quad f_{12} = -b. \end{aligned} \quad (3.55)$$

here  $a, b \in \mathbb{R}$  are arbitrary parameters. Now we can determine the expression  $Z_6$  i.e. the terms tri-linear in the ghost fields and of canonical dimension 5:

$$Z_6 = 4a u^\alpha(\partial^\mu u^\nu)(\partial_\mu \partial_\nu \tilde{u}_\alpha) \quad (3.56)$$

and the last equation of the system (3.46) admits a (non-unique) solution. One can choose the “minimal” solution:

$$S^\mu = 2a [u_\nu (\partial^\mu u_\rho) (\partial^\nu \tilde{u}^\rho) + u_\nu (\partial^\mu u_\rho) (\partial^\rho \tilde{u}^\nu) - u_\nu u_\rho (\partial^\mu \partial^\nu \tilde{u}^\rho)]. \quad (3.57)$$

Finally we consider the fifth equation (3.46). First we have the explicit expression:

$$\begin{aligned} Z_4 \equiv & -\frac{1}{2} \left[ c^{(4)} + m d_2^{(7)} \right] (\partial^\mu u^\nu) u_\mu \tilde{u}_\nu - \frac{1}{2} \left[ c^{(4)} + m d_3^{(7)} \right] (\partial^\nu u^\mu) u_\mu \tilde{u}_\nu \\ & + \frac{1}{2} \left[ \frac{1}{2} c^{(4)} + c^{(5)} - \frac{1}{4} m^2 c_2^{(11)} - \frac{1}{2} m^2 c_3^{(13)} - m d_1^{(7)} \right] (\partial^\nu u_\nu) u_\mu \tilde{u}^\mu \\ & + \frac{1}{2} \left[ m^2 c_2^{(11)} - m^2 c_3^{(11)} - m d_1^{(8)} + m d_3^{(8)} \right] u^\mu u^\nu (\partial_\mu \tilde{u}_\nu). \end{aligned} \quad (3.58)$$

It follows easily that we can take  $d_1 = 0$  in the expression (3.40) of  $\tilde{T}^\mu$ . Then we get from the fifth equation (3.46):

$$d_0 = -2m^2 a - \frac{3}{4} m^2 b, \quad (3.59)$$

so we get the result from the statement. ■

We remark that the basic idea of solving the gauge invariance problem was to use the equation (3.50) instead of the original gauge invariance condition (2.46); the equation (3.50) is simpler because we do not need an ansatz for  $T^\mu$ ; only an ansatz for  $Z^\mu$  is necessary.

The solution  $T^{(a)}$  gives in the massless limit the usual gravity theory [29] plus the new term  $4H^{\mu\nu}(\partial_\mu v_\rho)(\partial_\nu v^\rho)$ . The usual choice is  $a = -\frac{1}{4}$ .

One can re-express  $T^{(a)}$  using the variables from (2.13); we have

**Proposition 3.2** *In the variables*

$$h_{\mu\nu} \equiv H_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \Phi \quad h \equiv h^\mu_\mu = 2\Phi \quad (3.60)$$

*the expression  $T^{(a)}$  from the preceding theorem is equivalent to:*

$$\begin{aligned} T = & h^{\mu\nu} (\partial_\mu h) (\partial_\nu h) - 2h^{\mu\nu} (\partial_\mu h_{\rho\sigma}) (\partial_\nu h^{\rho\sigma}) - 4h_{\mu\nu} (\partial_\rho h^{\mu\sigma}) (\partial_\sigma h^{\nu\rho}) \\ & - 2h^{\mu\nu} (\partial_\rho h_{\mu\nu}) (\partial^\rho h) + 4h^{\mu\nu} (\partial_\sigma h_{\mu\rho}) (\partial^\sigma h^\rho_\nu) \\ & + 4(\partial_\mu h^{\mu\nu}) u^\rho (\partial_\rho \tilde{u}_\nu) - 4h^{\mu\nu} (\partial_\mu u^\rho) (\partial_\nu \tilde{u}_\rho) + 4h^{\mu\nu} (\partial_\mu v_\rho) (\partial_\nu v^\rho) \\ & - 4m (\partial_\mu v_\nu) u^\mu \tilde{u}^\nu + m^2 \left( -\frac{4}{3} h^{\mu\nu} h_{\mu\rho} h^\rho_\nu + h^{\mu\nu} h_{\mu\nu} h - \frac{1}{6} h^3 \right). \end{aligned} \quad (3.61)$$

*In these conditions one can take in (2.46)*

$$\begin{aligned} T_\mu^{(a)} = & u^\nu [ -(\partial_\mu h) (\partial_\nu h) + 2(\partial_\mu h_{\rho\sigma}) (\partial_\nu h^{\rho\sigma}) - 4(\partial_\rho h^{\rho\sigma}) (\partial_\nu h_{\mu\sigma}) \\ & + 4(\partial_\nu h^{\rho\sigma}) (\partial_\rho h_{\mu\sigma}) - 4(\partial_\mu v_\rho) (\partial_\nu v^\rho) ] \\ & + u_\mu \left[ \frac{1}{2} (\partial_\nu h) (\partial^\nu h) - (\partial_\nu h_{\rho\sigma}) (\partial^\nu h^{\rho\sigma}) - 2(\partial_\rho h_{\nu\sigma}) (\partial^\sigma h^{\nu\rho}) + 2(\partial_\nu v_\rho) (\partial^\nu v^\rho) \right] \end{aligned}$$

$$\begin{aligned}
& +(\partial^\rho u^\nu)[4h_{\rho\sigma}(\partial^\sigma h_{\mu\nu}) + 4h_{\mu\sigma}(\partial_\nu h_{\rho}^\sigma) + 2h_{\nu\rho}(\partial_\mu h) - 4h_{\rho\sigma}(\partial_\mu h_{\nu}^\sigma)] \\
& -4(\partial^\rho u_\mu)h^{\nu\sigma}(\partial_\nu h_{\rho\sigma}) + (\partial^\rho u_\rho)[-h(\partial_\mu h) + 2h^{\nu\sigma}(\partial_\mu h_{\nu\sigma})] \\
& +2[u^\nu(\partial_\mu u^\rho)(\partial_\nu \tilde{u}_\rho) + u^\nu(\partial_\nu u^\rho)(\partial_\mu \tilde{u}_\rho) - u_\mu(\partial^\nu u^\rho)(\partial_\nu \tilde{u}_\rho)] \\
& +m^2 u_\mu \left( h^{\nu\rho} h_{\nu\rho} - \frac{1}{2} h^2 \right). \tag{3.62}
\end{aligned}$$

**Proof:** It is convenient to start from the expression  $T$  above and make the substitution  $H_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Phi$   $h \equiv 2\Phi$ . Then one makes the transformations described at the beginning of the proof of the preceding theorem, namely we get rid of derivatives appearing on  $u_\rho$  subtraction total divergences. We also note that the fourth and the fifth terms from the expression of  $T$  can be eliminated if we use the identity (3.8) but have been included such that the linear approximation of the Hilbert Lagrangian is reproduced [29]. We obtain the expression  $T^{(a)}$  from the statement of the theorem.

The computation of  $T_\mu^{(a)}$  is not very difficult and provides an impressive check that the preceding computations are right. It is not hard to obtain by direct computation that

$$d_Q T = i[u_\mu t^\mu + (\partial_\nu u_\mu)t^{\mu\nu} + (\partial_\nu \partial_\rho u_\mu)t^{\mu\nu\rho} + s] \tag{3.63}$$

where the expressions  $t^\mu, t^{\mu\nu}, t^{\mu\nu\rho}$  are bi-linear in the fields  $h_{\mu\nu}, v_\mu$  and  $s$  is tri-linear in the ghost fields. We note that we have a certain freedom in choosing the expression  $t^{\mu\nu\rho}$  if we do not impose symmetry in  $\nu, \rho$ . Now, as in the proof of the preceding theorem, one makes “integrations by parts” and rewrites the preceding expressions as follows:

$$d_Q T = i(u^\mu y_\mu + \partial^\mu x_\mu + s) \tag{3.64}$$

where

$$y_\mu = t_\mu - \partial^\nu t_{\mu\nu} + \partial^\nu \partial^\rho t_{\mu\nu\rho}; \tag{3.65}$$

and

$$\begin{aligned}
x_\mu &= u^\nu x_{\mu\nu} + (\partial^\rho u^\nu) x_{\mu\nu\rho} \\
x_{\mu\nu} &\equiv t_{\nu\mu} - \partial^\rho t_{\nu\rho\mu} \quad x_{\mu\nu\rho} \equiv t_{\nu\mu\rho}.
\end{aligned} \tag{3.66}$$

By direct computation we can prove that

$$y_\mu = 0 \tag{3.67}$$

It remains to prove that the expression  $s_\mu$  from the statement is such that  $s = \partial^\mu s_\mu$  and the expression  $d_Q T$  above is exhibited as the total divergence from the statement. ■

In the same way one can also re-express  $T^{(b)}$  using the variables from (3.60); we have

**Proposition 3.3** *The expression  $T^{(b)}$  from the preceding theorem is equivalent to:*

$$T = -h^{\mu\nu}(\partial_\mu v_\rho)(\partial_\nu v^\rho) + 2h^{\mu\nu}(\partial_\mu v_\nu)(\partial^\rho v_\rho) - \frac{1}{2}v(\partial_\rho v^\rho)^2$$

$$\begin{aligned}
& -2h^{\mu\nu}(\partial_\mu v_\rho)(\partial^\rho v_\nu) + \frac{1}{2}h(\partial_\mu v_\rho)(\partial^\rho v^\mu) \\
& + m \left[ h^{\mu\nu}(\partial_\rho h_{\mu\nu})v^\rho - \frac{1}{2}h(\partial_\rho h)v^\rho - 2h^{\mu\nu}(\partial_\nu h_{\mu\rho})v^\rho + h_{\rho\nu}(\partial^\nu h)v^\rho \right. \\
& + \frac{1}{2}h(\partial^\mu h_{\mu\rho})v^\rho + (\partial_\rho v^\rho)u_\mu \tilde{u}^\mu - \frac{1}{2}(\partial_\mu v_\nu)u^\mu \tilde{u}^\nu + (\partial_\nu v_\mu)u^\mu \tilde{u}^\nu + v^\nu u^\mu (\partial_\nu \tilde{u}_\mu) \\
& \quad \left. + v_\mu u^\mu (\partial^\nu \tilde{u}_\nu) - \frac{1}{2}v^\nu u^\mu (\partial_\mu \tilde{u}_\nu) + 2v_\mu v_\nu (\partial^\mu v^\nu) \right] \\
& + m^2 \left( -\frac{1}{3}h^{\mu\nu}h_{\mu\rho}h_\nu{}^\rho + \frac{1}{4}h^{\mu\nu}h_{\mu\nu}h - \frac{1}{24}h^3 - h^{\mu\nu}u_\mu \tilde{u}_\nu \right. \\
& \quad \left. + \frac{1}{4}hu^\mu \tilde{u}_\mu + \frac{3}{2}h^{\mu\nu}v_\mu v_\nu - \frac{1}{4}hv^\mu v_\mu \right) \tag{3.68}
\end{aligned}$$

In these conditions one can take in (2.46)

$$\begin{aligned}
T_\mu^{(b)} = & u^\nu \left\{ (\partial_\mu v_\rho)(\partial_\nu v^\rho) - (\partial_\nu v_\mu)(\partial_\rho v^\rho) - \frac{1}{2}(\partial_\mu v_\nu)(\partial_\rho v^\rho) \right. \\
& + (\partial_\nu v_\rho)(\partial^\rho v_\mu) + \frac{1}{2}(\partial_\mu v_\rho)(\partial^\rho v_\nu) + \frac{1}{2}v_\nu(\partial_\mu \partial_\rho v^\rho) - \frac{1}{2}v^\rho(\partial_\mu \partial_\rho v_\nu) \\
& + m \left[ -(\partial^\rho h_{\mu\rho})v_\nu - h_{\mu\nu}(\partial^\rho v_\rho) + h_{\mu\rho}(\partial_\nu v^\rho) + h_{\nu\rho}(\partial^\rho v_\mu) - \frac{1}{2}v(\partial_\nu v_\mu) \right. \\
& \quad - (\partial_\rho h_{\mu\nu})v^\rho + (\partial_\nu h_{\mu\rho})v^\rho + \frac{1}{2}(\partial_\mu h_{\nu\rho})v^\rho - \frac{1}{4}(\partial_\mu h)v_\nu \\
& \quad \left. - \frac{1}{2}(\partial_\nu h)v_\mu + \frac{1}{4}h(\partial_\mu v_\nu) - \frac{1}{2}h_{\nu\rho}(\partial_\mu v^\rho) \right] \\
& \quad \left. + m^2 \left( -\frac{5}{2}v_\mu v_\nu + h_{\mu\rho}h_\nu{}^\rho - \frac{1}{2}h_{\mu\nu}h \right) \right\} \\
& + u_\mu \left\{ -\frac{1}{2}(\partial_\nu v_\rho)(\partial^\nu v^\rho) + \frac{1}{2}(\partial_\rho v_\rho)^2 - \frac{1}{2}(\partial_\nu v_\rho)(\partial^\rho v^\nu) \right. \\
& + m \left[ -h^{\rho\sigma}(\partial_\rho v_\sigma) + \frac{1}{2}h(\partial_\rho v^\rho) + \frac{1}{2}(\partial_\rho h)v^\rho - \frac{1}{2}(\partial^\rho h_{\rho\sigma})v^\sigma \right] \\
& \quad \left. + m^2 \left( -\frac{1}{4}h_{\rho\sigma}h^{\rho\sigma} + \frac{1}{8}h^2 + \frac{1}{2}v^\nu v_\nu \right) \right\} + m(\partial^\rho u^\nu)h_{\mu\rho}v_\nu \\
& - \frac{1}{2}(\partial_\mu u_\nu) \left[ v^\nu(\partial^\rho v_\rho) - v_\rho(\partial^\rho v^\nu) - m \left( h^{\nu\rho}v_\rho - \frac{1}{2}hv^\nu \right) \right] - \frac{3}{4}m^2 u_\mu u_\nu \tilde{u}^\nu. \tag{3.69}
\end{aligned}$$

## 4 Second Order Gauge Invariance

In this Section we consider the second order gauge invariance. For this we must construct the chronological products  $T(x, y)$  and  $T_\mu(x, y)$  such that the identity (3.39) is verified. The construction procedure is well-known: one computes first the corresponding causal commutators  $[T(x), T(y)]$  and  $[T_\mu(x), T(y)]$  and makes the substitution  $D_m(x - y) \mapsto D_m^F(x - y)$  i.e. one substitutes the causal Pauli-Jordan distribution by the corresponding Feynman propagator and similar substitutions for the loop graphs; one obtains the expressions  $T^F(x, y)$  and  $T_\mu^F(x, y)$  which verify all Bogoliubov axioms but might spoil second order gauge invariance. To restore it we must annihilate some anomalies and make finite renormalizations. These finite renormalizations must also preserve the power counting theorem which in this case says [29] that the expressions  $T(x, y)$  and  $T^\mu(x, y)$  should be of the form

$$T(x, y) = \sum_j t_j(x - y) W_j(x, y) \quad (4.1)$$

where  $W_j$  are Wick polynomials and  $t_j$  are distributions such that one has

$$\omega(t_j) + \deg(W_j) \leq 6. \quad (4.2)$$

The origin of the anomalies is explained in [12], [29]. One starts from the identity

$$d_Q[T(x), T(y)] = i \frac{\partial}{\partial x^\mu} [T^\mu(x), T(y)] + i \frac{\partial}{\partial y^\mu} [T(x), T^\mu(y)] \quad (4.3)$$

which follows from first order gauge invariance. If one substitutes expressions of the type (4.1) for the causal commutators

$$D(x, y) = \sum_j d_j(x - y) W_j(x, y) \quad (4.4)$$

then the preceding identity reduces to some identities verified by the distributions  $d_j$ . When we make the causal splitting of these distributions, preserving the degree of singularity, some of these identities are lost and we get anomalies. From tree Feynman graphs we get the identity

$$(\partial^2 + m^2) D_m = 0, \quad (4.5)$$

which cannot be split causally preserving the degree of singularity; indeed it is well known that

$$(\partial^2 + m^2) D_m^F = \delta(x - y). \quad (4.6)$$

From loop graphs we get the identities

$$\partial_\mu D_{m_1, m_2}^\mu = -m_2^2 D_{m_1, m_2} + \eta_{\mu\nu} D_{m_1, m_2}^{\mu\nu} \quad (4.7)$$

$$\partial_\mu D_{m_1, m_2}^{\mu\nu} = -m_1^2 D_{m_2, m_1}^\nu + \tilde{D}_{m_1, m_2}^\mu \quad (4.8)$$

$$\partial_\mu \tilde{D}_{m_1, m_2}^{\mu\nu} = -m_2^2 D_{m_1, m_2}^\nu + \tilde{D}_{m_1, m_2}^\mu \quad (4.9)$$

where we have introduced the following distributions with causal support:

$$\begin{aligned} D_{m_1, m_2} &\equiv D_{m_1}^{(+)} D_{m_2}^{(+)} - D_{m_1}^{(-)} D_{m_2}^{(-)} \\ D_{m_1, m_2}^\mu &\equiv D_{m_1}^{(+)} \partial^\mu D_{m_2}^{(+)} - D_{m_1}^{(-)} \partial^\mu D_{m_2}^{(-)} \\ D_{m_1, m_2}^{\mu\nu} &\equiv \partial^\mu D_{m_1}^{(+)} \partial^\nu D_{m_2}^{(+)} - \partial^\mu D_{m_1}^{(-)} \partial^\nu D_{m_2}^{(-)} \\ \tilde{D}_{m_1, m_2}^{\mu\nu} &\equiv D_{m_1}^{(+)} \partial^\mu \partial^\nu D_{m_2}^{(+)} - D_{m_1}^{(-)} \partial^\mu \partial^\nu D_{m_2}^{(-)} \\ \tilde{D}_{m_1, m_2}^\mu &\equiv \partial_\nu D_{m_1}^{(+)} \partial^\mu \partial^\nu D_{m_2}^{(+)} - \partial_\nu D_{m_1}^{(-)} \partial^\mu \partial^\nu D_{m_2}^{(-)}. \end{aligned} \quad (4.10)$$

It is easy to prove that one can use the arbitrariness of the causal splitting of these distributions such that one can eliminate all anomalies of the Ward identities (4.7) - (4.9). So it follows that only tree Feynman graphs can produce anomalies.

We apply this strategy to the quantum gravity model from the preceding Section.

First we consider the theory given by the interaction Lagrangian  $T = T^{(a)}$ .

**Theorem 4.1** *The Lagrangian  $T = T^{(a)}$  gives a theory gauge invariant in the second order of the perturbation theory if we perform convenient finite renormalizations of the second-order chronological products*

$$T(x, y) = T^F(x, y) + i \delta(x - y) N(x) \quad T_\mu(x, y) = T_\mu^F(x, y) + i \delta(x - y) N_\mu(x) \quad (4.11)$$

where  $N$  and  $N^\mu$  are some Wick polynomials. In particular:

$$\begin{aligned} N &= 16h^{\mu\nu} h^{\rho\sigma} (\partial_\rho h_{\mu\nu}) (\partial_\sigma h) - 8h^{\mu\nu} h^{\rho\sigma} (\partial_\alpha h_{\mu\nu}) (\partial^\alpha h_{\rho\sigma}) - 32h^{\mu\nu} h_{\nu\rho} (\partial^\alpha h^{\rho\beta}) (\partial_\beta h_{\mu\alpha}) \\ &\quad - 32h^{\mu\nu} h^{\rho\sigma} (\partial_\mu h^{\rho\alpha}) (\partial_\nu h_\sigma{}^\alpha) + 32h^{\mu\nu} h_{\nu\rho} (\partial^\alpha h_{\mu\beta}) (\partial_\alpha h^{\rho\beta}) + 16h^{\mu\nu} h^{\rho\sigma} (\partial_\alpha h_{\mu\rho}) (\partial^\alpha h_{\nu\sigma}) \\ &\quad - 16h^{\mu\nu} h_{\nu\rho} (\partial^\alpha h_\mu{}^\rho) (\partial_\alpha h) + 16u^\rho (\partial_\rho \tilde{u}^\nu) u^\sigma (\partial_\sigma \tilde{u}_\nu) \\ &\quad + 2m^2 \left( \frac{1}{12} h^4 - h^{\mu\nu} h_{\mu\nu} h^2 + \frac{8}{3} h^{\mu\nu} h_{\nu\rho} h_\mu{}^\rho h + h^{\mu\nu} h_{\mu\nu} h^{\rho\sigma} h_{\rho\sigma} - 4h^{\mu\nu} h_{\nu\rho} h_{\mu\sigma} h^{\rho\sigma} \right). \end{aligned} \quad (4.12)$$

**Proof:** As it is known, the first step is to compute the causal commutator  $[T^\mu(x), T(y)]$ . The anomalies are produced by two types of terms in  $T^\mu(x)$ : (a) with the index  $\mu$  appearing in a derivative  $\partial_\mu$ ; (b) with the index  $\mu$  appearing in the combination  $h_{\mu\rho}$ . Inspecting the expression from prop. 3.2 we have

$$\begin{aligned} T_\mu &= T_1(\partial_\mu h) + T_2^{\alpha\beta}(\partial_\mu h_{\alpha\beta}) + T_3^\nu(\partial_\mu \tilde{u}_\nu) + T_4^\nu(\partial_\mu u_\nu) + T_5^\nu(\partial_\mu v_\nu) \\ &\quad + S^{\nu\rho}(\partial_\nu h_{\mu\rho}) + S^\rho h_{\mu\rho} + \dots \end{aligned} \quad (4.13)$$

where by  $\dots$  we mean terms which do not produce anomalies and we have defined:

$$\begin{aligned} T_1 &\equiv -u^\nu (\partial_\nu h) - (\partial^\rho u_\rho) h + 2(\partial^\rho u^\nu) h_{\nu\rho} \\ T_2^{\alpha\beta} &\equiv 2 \left[ u_\lambda (\partial^\lambda h^{\alpha\beta}) + (\partial^\lambda u_\lambda) h^{\alpha\beta} - (\partial_\lambda u^\alpha) h^{\lambda\beta} - (\partial_\lambda u^\beta) h^{\lambda\alpha} \right] \\ T_3^\nu &\equiv 2u_\rho (\partial^\rho u^\nu) \end{aligned}$$

$$\begin{aligned}
T_4^\nu &\equiv -2u_\rho(\partial^\rho \tilde{u}^\nu) \\
T_5^\nu &\equiv -4u_\rho(\partial^\rho v^\nu) \\
S^{\nu\rho} &= 4 \left[ -u^\nu(\partial_\sigma h^{\rho\sigma}) + u_\sigma(\partial^\sigma h^{\nu\rho}) + (\partial_\sigma u^\rho)h^{\nu\sigma} \right] \\
S^\rho &= 4(\partial_\sigma u_\lambda)(\partial^\lambda h^{\rho\sigma});
\end{aligned} \tag{4.14}$$

here we have imposed the symmetry condition

$$T_2^{\rho\sigma} = (\rho \leftrightarrow \sigma). \tag{4.15}$$

It is also convenient to denote

$$T_2 \equiv T_{2;\alpha\beta}\eta^{\alpha\beta} \tag{4.16}$$

with the explicit expression

$$T_2 = -2T_1. \tag{4.17}$$

One has to compute the commutator of  $\partial^\mu h, \partial^\mu h^{\alpha\beta}, \partial^\mu \tilde{u}^\rho, \partial^\mu u^\rho, \partial^\mu A^\rho$  with the 12 linear independent Wick monomials which appear in the expression of the total coupling  $T$ . Using the causal (anti)commutation relations from Section 4 we get :

$$\begin{aligned}
[\partial_\mu h(x), T(y)] &= i \partial_\mu D_m(x-y)D_1(y) + i \partial_\mu \partial_\nu D_m(x-y)D_1^\nu(y) \\
[\partial_\mu H^{\alpha\beta}(x), T(y)] &= i \partial_\mu D_m(x-y)D_2^{\alpha\beta}(y) + i \partial_\mu \partial_\nu D_m(x-y)D_2^{\alpha\beta;\nu}(y) \\
[\partial_\mu \tilde{u}^\rho(x), T(y)] &= i \partial_\mu D_m(x-y)D_3^\rho(y) + i \partial_\mu \partial_\nu D_m(x-y)D_3^{\rho;\nu}(y) \\
[\partial_\mu u^\rho(x), T(y)] &= i \partial_\mu D_m(x-y)D_4^\rho(y) + i \partial_\mu \partial_\nu D_m(x-y)D_4^{\rho;\nu}(y) \\
[\partial_\mu v^\rho(x), T(y)] &= i \partial_\mu D_m(x-y)D_5^\rho(y) + i \partial_\mu \partial_\nu D_m(x-y)D_5^{\rho;\nu}(y)
\end{aligned} \tag{4.18}$$

where

$$\begin{aligned}
D_1 &= -(\partial^\rho h)(\partial_\rho h) + 2(\partial^\rho h^{\alpha\beta})(\partial_\rho h_{\alpha\beta}) - 4(\partial^\alpha h^{\beta\lambda})(\partial_\beta h_{\alpha\lambda}) \\
&\quad - 4(\partial^\rho u^\sigma)(\partial_\rho \tilde{u}_\sigma) + 4a (\partial^\rho v^\sigma)(\partial_\rho v_\sigma)
\end{aligned} \tag{4.19}$$

$$D_1^\nu = -4h^{\nu\rho}(\partial_\rho h) + 8h_{\rho\sigma}(\partial^\rho h^{\nu\sigma}) + 2h(\partial^\nu h_\sigma) - 4u_\rho(\partial^\rho \tilde{u}^\nu). \tag{4.20}$$

$$D_2^{\alpha\beta} = \mathcal{D}_2^{\alpha\beta} + D_2\eta^{\alpha\beta}, \tag{4.21}$$

where

$$\begin{aligned}
\mathcal{D}_2^{\alpha\beta} &= -(\partial^\alpha h)(\partial^\beta h) + 2(\partial^\alpha h^{\sigma\lambda})(\partial^\beta h_{\sigma\lambda}) + 4(\partial_\rho h^{\sigma\alpha})(\partial_\sigma h^{\rho\beta}) \\
&\quad + 2(\partial^\rho h^{\alpha\beta})(\partial_\rho h) - 4(\partial^\rho h^{\sigma\alpha})(\partial_\rho h_{\sigma}^\beta) + 2(\partial^\alpha u_\rho)(\partial^\beta \tilde{u}^\rho) + 2(\partial^\beta u_\rho)(\partial^\alpha \tilde{u}^\rho) \\
&\quad - 4(\partial^\alpha v_\rho)(\partial^\beta v^\rho) + 2m^2 \left( 2v^{\rho\alpha}h_{\rho}^\beta - v^{\alpha\beta}h \right)
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}
D_2 &= -\frac{1}{2}(\partial^\rho h)(\partial_\rho h) + (\partial^\rho h^{\alpha\beta})(\partial_\rho h_{\alpha\beta}) - 2(\partial^\alpha h^{\beta\lambda})(\partial_\beta h_{\alpha\lambda}) - 2(\partial^\rho u^\sigma)(\partial_\rho \tilde{u}_\sigma) \\
&\quad + 2(\partial^\rho v^\sigma)(\partial_\rho v_\sigma) + m^2 \left( -h^{\alpha\beta}h_{\alpha\beta} + \frac{1}{2}h^2 \right),
\end{aligned} \tag{4.23}$$



$$D_2^{\alpha\beta;\nu} = \mathcal{D}_2^{\alpha\beta;\nu} + \eta^{\alpha\beta} D_2^\nu \quad (4.24)$$

with

$$\begin{aligned} \mathcal{D}_2^{\alpha\beta;\nu} = & -4(\partial_\rho h^{\alpha\beta})h^{\nu\rho} - 4(\partial^\alpha h^{\nu\sigma})h^\beta \cdot \sigma - 4(\partial^\beta h^{\nu\sigma})h^\alpha \cdot \sigma \\ & - 2h^{\alpha\beta}(\partial^\nu h) + 4h^{\alpha\rho}(\partial^\nu h_{\cdot\rho}^\beta) + 4h^{\beta\rho}(\partial^\nu h_{\cdot\rho}^\alpha) + 2\eta^{\alpha\nu}u_\rho(\partial^\rho \tilde{u}^\beta) + 2\eta^{\beta\nu}u_\rho(\partial^\rho \tilde{u}^\alpha) \end{aligned} \quad (4.25)$$

$$D_2^\nu = 4(\partial^\rho h^{\nu\sigma})h_{\rho\sigma} + h(\partial^\nu h) - 2h_{\rho\sigma}(\partial^\nu h^{\rho\sigma}) - 2u_\rho(\partial^\rho \tilde{u}^\nu) \quad (4.26)$$

$$D_3^\rho = -4\left[(\partial^\sigma h_{\lambda\sigma})(\partial^\rho \tilde{u}^\lambda) - m(\partial^\rho v^\sigma)\tilde{u}^\sigma\right] \quad (4.27)$$

$$D_3^{\rho;\nu} = -4h^{\nu\sigma}(\partial_\sigma \tilde{u}^\rho) \quad (4.28)$$

$$D_4^\rho \equiv 4m(\partial^\sigma v^\rho)u_\sigma \quad (4.29)$$

$$D_4^{\rho;\nu} \equiv 4\left[(\partial_\sigma h^{\rho\sigma})u^\nu - h^{\nu\sigma}(\partial_\sigma u^\rho)\right] \quad (4.30)$$

$$D_5^\rho \equiv 0 \quad (4.31)$$

$$D_5^{\rho;\nu} \equiv 2\left[-2h^{\nu\sigma}(\partial_\sigma v^\rho) + mu^\nu \tilde{u}^\rho\right]. \quad (4.32)$$

One also has:

$$\begin{aligned} [h_{\mu\rho}(x), T(y)] &= i \partial_\mu D_m(x-y)E_\rho(y) + \dots \\ [\partial_\nu h_{\mu\rho}(x), T(y)] &= i \partial_\mu \partial_\nu D_m(x-y)E_\rho(y) + \dots \end{aligned} \quad (4.33)$$

with

$$E_\rho = 2u^\sigma(\partial_\sigma \tilde{u}_\rho). \quad (4.34)$$

From these formulæ we obtain

$$[T^\mu(x), T(y)] = i \partial_\mu D_m(x-y)A(x, y) + i \partial_\mu \partial_\nu D_m(x-y)A^\nu(x, y) + \dots \quad (4.35)$$

where

$$A(x, y) \equiv T_1(x)\mathcal{D}_1(y) + T_2^{\alpha\beta}(x)\mathcal{D}_{2;\alpha\beta}(y) + \sum_{j=3}^5 T_j^\nu(x)D_{j;\nu}(y) + S^\rho(x)E_\rho(y) \quad (4.36)$$

and

$$A^\nu(x, y) \equiv T_1(x)\mathcal{D}_1^\nu(y) + T_{2;\alpha\beta}(x)\mathcal{D}_2^{\alpha\beta;\nu}(y) + \sum_{j=3}^5 T_{j;\rho}(x)D_j^{\rho;\nu}(y) + S^{\nu\rho}(x)E_\rho(y) \quad (4.37)$$

where

$$\begin{aligned}\mathcal{D}_1 &\equiv D_1 - 2D_2 \\ \mathcal{D}_1^\nu &\equiv D_1^\nu - 2D_2^\nu.\end{aligned}\tag{4.38}$$

From formula (4.35) we get the anomaly

$$a(x, y) = \delta(x - y)A(x) + [\partial_\nu \delta(x - y)]A^\nu(x, y)\tag{4.39}$$

By integration by parts we can rewrite this as follows:

$$a(x, y) = \delta(x - y)\mathcal{A}(x) - \partial_\nu^y \left[ \delta(x - y)\mathcal{A}^\nu(y) \right]\tag{4.40}$$

with

$$\begin{aligned}\mathcal{A}(x) &\equiv A(x) + \partial_\nu^y A^\nu(x, y)|_{y=x} \\ \mathcal{A}^\nu(x) &\equiv A^\nu(x, x)\end{aligned}\tag{4.41}$$

From the preceding formulæ we get

$$\mathcal{A} = T_1 \tilde{\mathcal{D}}_1 + T_{2;\alpha\beta} \tilde{\mathcal{D}}_2^{\alpha\beta} + \sum_{j=3}^5 T_{j;\rho} \tilde{D}_j^\rho + S^\rho E_\rho + S^{\nu\rho} (\partial_\nu E_\rho)\tag{4.42}$$

where

$$\begin{aligned}\tilde{\mathcal{D}}_j &\equiv \mathcal{D}_j + \partial_\nu \mathcal{D}_j^\nu \quad j = 1, 2 \\ \tilde{D}_j &\equiv D_j + \partial_\nu D_j^\nu \quad j = 3, 4, 5, 6\end{aligned}\tag{4.43}$$

The total anomaly comes from the two commutators  $[T^\mu(x), T(y)] + [T^\mu(y), T(x)]$  and is

$$\mathcal{A}(x, y) = a(x, y) + a(y, x).\tag{4.44}$$

Now, the second-order gauge invariance condition (2.45) for  $n = 2$  is fulfilled *iff* one can write this anomaly as follows:

$$\mathcal{A}(x, y) = 2id_Q N(x, y) + \partial_\mu^x N^\mu(x, y) + \partial_\mu^y N^\mu(y, x)\tag{4.45}$$

where  $N(x, y)$  and  $N^\mu(x, y)$  are quasi-local Wick polynomials. One can show rather easily that this condition is equivalent to:

$$\mathcal{A} = id_Q N + \partial_\mu N^\mu\tag{4.46}$$

for some Wick polynomials (in one variable)  $N$  and  $N^\mu$  of ghost number 0 and resp. 1; here  $\mathcal{A}$  is given by the first formula (4.41). From power counting considerations (4.2) one also has the limitations

$$\deg(N), \deg(N^\mu) \leq 6.\tag{4.47}$$

The condition (4.46) is the basic condition which we will investigate from now on; it splits in distinct conditions for different sectors.

a) We consider first the part  $\mathcal{A}_{uhhh}$  of the anomaly which is linear in  $u_\mu$  and tri-linear in the field  $h_{\alpha\beta}$ . From the expression of the anomaly we have

$$\mathcal{A}_{uhhh} = u^\mu \mathbf{A}_\mu + (\partial_\mu u^\mu) \mathbf{A} + (\partial^\nu u^\mu) \mathbf{A}_{\mu\nu} \quad (4.48)$$

where

$$\begin{aligned} \mathbf{A}_\mu &\equiv -(\partial_\mu h) \tilde{\mathcal{D}}_1 + 2(\partial_\mu h^{\alpha\beta}) \tilde{\mathcal{D}}_{2;\alpha\beta}^{hh} \\ \mathbf{A} &\equiv -h \tilde{\mathcal{D}}_1 + 2h^{\alpha\beta} \tilde{\mathcal{D}}_{2;\alpha\beta}^{hh} \\ \mathbf{A}_{\mu\nu} &\equiv 2h_{\mu\nu} \tilde{\mathcal{D}}_1 - 4h_{\mu}^{\rho} \tilde{\mathcal{D}}_{2;\nu\rho}^{hh}. \end{aligned} \quad (4.49)$$

By the symbol  $hh$  as an index means that we consider only the bi-linear part in  $h_{\alpha\beta}$  of the corresponding expression. We want to find out if it is possible to write

$$\mathcal{A}_{uhhh} = id_Q N_1 + \partial_\mu N_1^\mu \quad (4.50)$$

for some Wick polynomials  $N_1$  and  $N_1^\mu$  of ghost numbers 0 and resp. 1. We will not consider the generic form of  $N_1$  and  $\tilde{N}_1^\mu$ ; instead, we make the following ansatz

$$\tilde{N}_1^\mu = u_\nu t^{\mu\nu} + (\partial_\rho u_\nu) t^{\mu\nu\rho} \quad (4.51)$$

where the expressions  $t^{\dots}$  are tri-linear in the fields  $h_{\alpha\beta}$ , have null ghost number and are limited by the power counting conditions

$$\deg(t^{\mu\nu}) \leq 5, \quad \deg(t^{\mu\nu\rho}) \leq 4; \quad (4.52)$$

we will also suppose that the expression  $t^{\mu\nu\rho}$  does not contain terms with the factor  $\eta^{\mu\rho}$ .

If we consider in  $N_1$  only terms of the type  $hh(\partial h)(\partial h)$  and  $hhhh$ , we have the following generic expressions of  $d_Q N_1$ :

$$i d_Q N_1 = (\partial^\nu u^\mu) \mathcal{B}_{\mu\nu} + (\partial^\nu \partial^\rho u^\mu) \mathcal{B}_{\mu\nu\rho} \quad (4.53)$$

where the expressions  $\mathcal{B}...$  are tri-linear in the fields  $h_{\alpha\beta}$ , have null ghost number and are limited by the power counting conditions

$$\deg(\mathcal{B}_{\mu\nu}) \leq 5, \quad \deg(\mathcal{B}_{\mu\nu\rho}) \leq 4; \quad (4.54)$$

we can suppose that the expression  $\mathcal{B}^{\mu\nu\rho}$  is symmetric in  $\nu, \rho$ .

The explicit expressions for  $\mathcal{B}...$  follow from the generic ansatz for  $N_1$ ; they depend on some unknown coefficients which are to be determined. We substitute everything in the equation (4.50) and get the following system:

$$\begin{aligned} \mathbf{A}^\mu &= \partial_\nu t^{\nu\mu} \\ \mathbf{A}^{\mu\nu} + \eta^{\mu\nu} \mathbf{A} &= \mathcal{B}^{\mu\nu} + t^{\nu\mu} + \partial_\rho t^{\rho\mu\nu} \\ \mathcal{B}^{\mu\nu\rho} + \mathcal{S}_{\nu\rho} t^{\nu\mu\rho} &= 0 \end{aligned} \quad (4.55)$$

From the last equation we can determine  $t^{\mu\nu\rho}$  (if we suppose that it is symmetric in  $\mu$  and  $\rho$ ). We substitute in the second equation of the system and determine  $t^{\mu\nu}$ . Finally we substitute this in the first equation of the system; it turns out that the system is consistent *iff* the following equation is true:

$$\mathbf{A}^\mu - \partial_\nu \mathbf{A}^{\mu\nu} - \partial^\mu \mathbf{A} = -\partial_\nu \mathcal{B}^{\mu\nu} + \partial_\nu \partial_\rho \mathcal{B}^{\mu\nu\rho}. \quad (4.56)$$

This is the basic equation which we now analyze. The expressions from the left hand side can be computed explicitly from the formulæ (4.49). The general strategy is to make an ansatz for  $N_1$ , compute the expressions  $\mathcal{B}_{\dots}$  and compute the right hand side of the equation. Then we get a system for the unknown coefficients of the finite renormalization  $N_1$ .

The ansatz for  $N_1$  can be guessed analyzing different contributions appearing in  $\mathcal{A}_{uhhh}$  and it is

$$\begin{aligned} N_1 = & f_1 h^{\mu\nu} h^{\rho\sigma} (\partial_\rho h_{\mu\nu}) (\partial_\sigma h) + f_2 h^{\mu\nu} h^{\rho\sigma} (\partial_\alpha h_{\mu\nu}) (\partial^\alpha h_{\rho\sigma}) \\ & + f_3 h^{\mu\nu} h_{\nu\rho} (\partial^\alpha h^{\rho\beta}) (\partial_\beta h_{\mu\alpha}) + f_4 h^{\mu\nu} h^{\rho\sigma} (\partial_\mu h^{\rho\alpha}) (\partial_\nu h_\sigma{}^\alpha) \\ & + f_5 h^{\mu\nu} h_{\nu\rho} (\partial^\alpha h_{\mu\beta}) (\partial_\alpha h^{\rho\beta}) + f_6 h^{\mu\nu} h^{\rho\sigma} (\partial_\alpha h_{\mu\rho}) (\partial^\alpha h_{\nu\sigma}) \\ & + f_7 h^{\mu\nu} h_{\nu\rho} (\partial^\alpha h_\mu{}^\rho) (\partial_\alpha h) + f_8 h^4 + f_9 h^{\mu\nu} h_{\mu\nu} h^2 \\ & + f_{10} h^{\mu\nu} h_{\nu\rho} h^\rho{}_{,\mu} H + f_{11} h^{\mu\nu} h_{\mu\nu} h^{\rho\sigma} h_{\rho\sigma} + f_{12} h^{\mu\nu} h_{\nu\rho} h_{\mu\sigma} h^{\rho\sigma}; \end{aligned} \quad (4.57)$$

after performing the computation of  $d_Q N_1$  we see that the expression  $\mathcal{B}_{\mu\nu\rho}$  does not have terms with the factor  $\eta_{\nu\rho}$  so  $t_{\mu\nu\rho} = -\mathcal{B}_{\nu\mu\rho}$  does not have terms with the factor  $\eta_{\mu\rho}$ ; this is consistent with the ansatz we have made for  $t_{\mu\nu\rho}$ . The consistency equation (4.56) gives after long but straightforward computations:

$$\begin{aligned} f_1 = 8 \quad f_2 = -4 \quad f_3 = f_4 = -16 \quad f_5 = 16 \quad f_6 = 8 \quad f_7 = -8 \\ f_8 = \frac{1}{12} m^2 \quad f_9 = -m^2 \quad f_{10} = \frac{8}{3} m^2 \quad f_{11} = m^2 \quad f_{12} = -4 m^2. \end{aligned} \quad (4.58)$$

b) We consider now the contribution  $\mathcal{A}_{uu\tilde{u}h}$  of the anomaly which is tri-linear in the ghost fields and linear in the field  $h_{\alpha\beta}$ . The explicit expression is:

$$\begin{aligned} \mathcal{A}_{uu\tilde{u}h} = & 8 \left[ u_\lambda (\partial^\lambda h^{\alpha\beta}) + (\partial^\lambda u_\lambda) h^{\alpha\beta} - (\partial_\lambda u^\alpha) h^{\lambda\beta} - (\partial_\lambda u^\beta) h^{\lambda\alpha} \right] \\ & \times \left[ (\partial_\alpha u_\rho) (\partial_\beta \tilde{u}^\rho) + (\partial_\alpha u_\rho) (\partial^\rho \tilde{u}_\beta) + u^\rho (\partial_\alpha \partial_\rho \tilde{u}_\beta) \right] \\ & - 8 u_\lambda (\partial^\lambda u^\nu) \left[ (\partial_\sigma h^{\rho\sigma}) (\partial_\nu \tilde{u}_\rho) + (\partial_\sigma h^{\rho\sigma}) (\partial_\rho \tilde{u}_\nu) + h^{\rho\sigma} (\partial_\rho \partial_\sigma \tilde{u}_\nu) \right] \\ & - 8 u_\lambda (\partial^\lambda \tilde{u}^\nu) \left[ (\partial^\rho \partial^\sigma h_{\nu\sigma}) u_\rho + (\partial^\sigma h_{\nu\sigma}) (\partial_\rho u^\rho) - (\partial_\rho h^{\rho\sigma}) (\partial_\sigma u_\nu) - h^{\rho\sigma} (\partial_\rho \partial_\sigma u_\nu) \right] \\ & + 8 (\partial_\nu u_\lambda) (\partial^\lambda h^{\nu\rho}) u^\sigma (\partial_\sigma \tilde{u}_\rho) \\ & + 8 \left[ -u^\nu (\partial_\lambda h^{\rho\lambda}) + u_\lambda (\partial^\lambda h^{\nu\rho}) + (\partial_\lambda u^\rho) h^{\nu\lambda} \right] \times \left[ (\partial_\nu u^\sigma) (\partial_\sigma \tilde{u}_\rho) + u^\sigma (\partial_\nu \partial_\sigma \tilde{u}_\rho) \right]. \end{aligned} \quad (4.59)$$

We want to write this expression in the form

$$\mathcal{A}_{uu\tilde{u}h} = i(d_Q N_2)_{uu\tilde{u}h} + \partial_\mu N_2^\mu. \quad (4.60)$$

We make the ansatz

$$N_2 = f_{13} u^\rho (\partial_\rho \tilde{u}^\nu) u^\sigma (\partial_\sigma \tilde{u}_\nu). \quad (4.61)$$

It is convenient to re-write the preceding formula (using “partial integration”) as follows:

$$\mathcal{A}_{uu\tilde{u}h} - i(d_Q N_2)_{uu\tilde{u}h} = \partial_\mu N_2^\mu + h_{\alpha\beta} U^{\alpha\beta} \quad (4.62)$$

where  $U^{\alpha\beta}$  is tri-linear in the ghost fields. One can obtain after some computations that one can fix the constant  $f_{13}$  such that  $U^{\alpha\beta} = 0$ ; namely we must have

$$f_{13} = 16. \quad (4.63)$$

c) The contribution linear in the field  $v^\mu$  of the anomaly is

$$\mathcal{A}_v = 8m \left[ u^\sigma (\partial_\sigma u_\nu) + (\partial_\sigma u^\sigma) u_\nu \right] (\partial^\nu v^\rho) \tilde{u}_\rho. \quad (4.64)$$

We have to subtract from this expression the contribution linear in  $v_\mu$  from  $d_Q N_2$ . The result is a divergence:

$$\mathcal{A}_v = i(d_Q N_2)_v + \partial_\mu N_3^\mu \quad N_3^\mu \equiv 8m u^\mu u^\nu (\partial_\nu v_\rho) \tilde{u}^\rho. \quad (4.65)$$

d) Now we consider the contribution bi-linear in the field  $v_\mu$ ; we have

$$\begin{aligned} \mathcal{A}_{vv} = & -8 \left[ u_\rho (\partial^\rho h^{\alpha\beta}) + (\partial^\rho u_\rho) h^{\alpha\beta} - (\partial_\rho u^\alpha) h^{\rho\beta} - (\partial_\rho u^\beta) h^{\rho\alpha} \right] (\partial_\alpha v_\sigma) (\partial_\beta v^\sigma) \\ & + 16 u_\rho (\partial^\rho v^\nu) \left[ (\partial_\lambda h^{\lambda\sigma}) (\partial_\sigma v_\nu) + h^{\lambda\sigma} (\partial_\lambda \partial_\sigma v_\nu) \right]. \end{aligned} \quad (4.66)$$

One can write this expression as a total divergence:

$$\begin{aligned} \mathcal{A}_{vv} = & \partial_\mu N_4^\mu \\ N_4^\mu \equiv & -8 \left[ u^\mu h^{\alpha\beta} (\partial_\alpha v_\rho) (\partial_\beta v^\rho) - 2 u_\rho h^{\mu\nu} (\partial^\rho v^\sigma) (\partial_\nu v_\sigma) \right]. \end{aligned} \quad (4.67)$$

This finished the proof of the theorem. ■

We now consider the general case (i.e.  $b \neq 0$ ). It is sufficient to consider the first two sectors. The sector  $uhhhh$  modifies only the values of the coefficients  $f_j$ . The anomaly  $\mathcal{A}_{uhhh}$  acquires the following expression: in (4.48) + (4.49) we have

$$\begin{aligned} \tilde{\mathcal{D}}_1 = & \dots + \frac{1}{2} m^2 b \left( h_{\rho\sigma} h^{\rho\sigma} - \frac{1}{2} h^2 \right) \\ \tilde{\mathcal{D}}_2^{\alpha\beta; HH} = & \dots + m^2 b \left( h^{\alpha\rho} h_\rho{}^{\cdot\beta} - \frac{1}{2} h^{\alpha\beta} h \right) \end{aligned} \quad (4.68)$$

where by  $\dots$  we mean the corresponding expressions from the case  $a = 1, b = 0$  multiplied by  $a^2$ . If we perform the same computations as in the preceding theorem we obtain the new values of the coefficients  $f_j$ :

$$\begin{aligned} f_1 = 8a^2 \quad f_2 = -4a^2 \quad f_3 = f_4 = -16a^2 \quad f_5 = 16a^2 \quad f_6 = 8a^2 \quad f_7 = -8a^2 \\ f_8 = \frac{1}{12} \left( a^2 + \frac{1}{4} ab \right) m^2 \quad f_9 = - \left( a^2 + \frac{1}{4} ab \right) m^2 \quad f_{10} = \frac{8}{3} \left( a^2 + \frac{1}{4} ab \right) m^2 \\ f_{11} = \left( a^2 + \frac{1}{4} ab \right) m^2 \quad f_{12} = -4 \left( a^2 + \frac{1}{4} ab \right) m^2. \end{aligned} \quad (4.69)$$

The sector  $uu\tilde{u}h$  gives the following expression of the anomaly:

$$\mathcal{A}_{uu\tilde{u}h} = \cdots + m^2 ab \mathcal{A}_1 + \frac{1}{16} m^2 b^2 \mathcal{A}_2 \quad (4.70)$$

where

$$\begin{aligned} \mathcal{A}_1 \equiv & \frac{1}{2} \left[ -u_\lambda (\partial^\lambda h) - (\partial^\lambda u_\lambda) h + 2(\partial_\lambda u_\nu) h^{\lambda\nu} \right] u^\rho \tilde{u}_\rho \\ & + 2 \left[ u_\lambda (\partial^\lambda h^{\alpha\beta}) + (\partial^\lambda u_\lambda) h^{\alpha\beta} - (\partial_\lambda u^\beta) h^{\lambda\alpha} - (\partial_\lambda u^\alpha) h^{\lambda\beta} \right] u_\alpha \tilde{u}_\beta \\ & + 2u_\lambda (\partial^\lambda u^\rho) \left( h_{\rho\sigma} \tilde{u}^\sigma - \frac{1}{4} h \tilde{u}^\rho \right) - 2u_\lambda (\partial^\lambda \tilde{u}^\rho) \left( h_{\rho\sigma} u^\sigma - \frac{1}{4} h u^\rho \right), \end{aligned} \quad (4.71)$$

$$\begin{aligned} \mathcal{A}_2 \equiv & h \left[ u^\rho (\partial^\lambda u^\lambda) \tilde{u}_\rho - 2u^\rho (\partial_\lambda u_\rho) \tilde{u}^\lambda - 2u^\rho (\partial_\rho u_\lambda) \tilde{u}^\lambda \right] \\ & - 2h^{\rho\sigma} \left[ u_\sigma (\partial^\lambda u^\lambda) \tilde{u}_\rho - 2u_\sigma (\partial_\lambda u_\rho) \tilde{u}^\lambda - 2u_\sigma (\partial_\rho u_\lambda) \tilde{u}^\lambda \right], \end{aligned} \quad (4.72)$$

where by  $\cdots$  we mean the expression (4.59) multiplied by  $a^2$ . If we try to write the total anomaly as a total divergence + coboundary, we obtain in the end:  $b = 0$ .

## 5 Conclusions

The evidence for a dark energy in our Universe motivates the introduction of the cosmological constant  $\Lambda$  into Einstein's equations. The corresponding Einstein - Hilbert Lagrangian is given by

$$L_E = -\frac{2}{\kappa^2}\sqrt{-g}(R - 2\Lambda), \quad \kappa^2 = 32\pi G \quad (5.1)$$

where  $G$  is Newton's constant and  $g = \det(g_{\mu\nu})$ . We want to expand this in powers of  $\kappa$ . For this purpose it is convenient to use the so-called Goldberg variables

$$\tilde{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}, \quad \tilde{g}_{\mu\nu} = (-g)^{-1/2}g_{\mu\nu}. \quad (5.2)$$

Now we write this metric tensor as

$$\tilde{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu} \quad (5.3)$$

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \eta_{\mu\nu} - \kappa h_{\mu\nu} + \kappa^2 h_{\mu\alpha} h_{\nu}^{\alpha} - \kappa^3 h_{\mu\alpha} h^{\alpha\beta} h_{\beta\nu} + \dots \\ &= \eta_{\mu\lambda}(\delta_{\nu}^{\lambda} - \kappa h_{\nu}^{\lambda} + \kappa^2 h_{\alpha}^{\lambda} h_{\nu}^{\alpha} - \kappa^3 h_{\alpha}^{\lambda} h^{\alpha\beta} h_{\beta\nu} + \dots) \end{aligned} \quad (5.4)$$

Then we can expand the determinant  $g$  and the Ricci scalar  $R$  in (1) in powers of  $\kappa$ ; details can be found in [29], Sect.5.5.

The first four orders in  $\kappa$ , i.e.  $O(\kappa^{-2}), \dots O(\kappa^2)$  come out to be:

$$\begin{aligned} L_E &= \frac{4\Lambda}{\kappa^2} + \frac{2}{\kappa}(\partial^2 h - \partial_{\alpha}\partial_{\beta}h^{\alpha\beta} + \Lambda h) \\ &+ \frac{1}{2}(\partial_{\gamma}h_{\alpha\beta})(\partial^{\gamma}h^{\alpha\beta}) - \frac{1}{2}\partial_{\gamma}h(\partial^{\gamma}h) + (\partial_{\alpha}h^{\alpha\beta})(\partial_{\beta}h) - (\partial_{\gamma}h^{\alpha\beta})(\partial_{\alpha}h_{\beta}^{\gamma}) \\ &+ \Lambda\left(\frac{1}{2}h^2 - h^{\alpha\beta}h_{\alpha\beta}\right) \\ &+ \kappa\left[L^{(1)} + 4\Lambda\left(\frac{1}{6}h^{\alpha\beta}h_{\beta\gamma}h_{\alpha}^{\gamma} - \frac{1}{8}h^{\alpha\beta}h_{\alpha\beta}h + \frac{1}{48}h^3\right)\right] \\ &+ \kappa^2\left\{L^{(2)} + 4\Lambda\left[\frac{1}{32}\left(h^{\alpha\beta}h_{\alpha\beta}\right)^2 + \frac{1}{12}hh^{\alpha\beta}h_{\beta\gamma}h_{\alpha}^{\gamma} - \frac{1}{32}h^2h^{\alpha\beta}h_{\alpha\beta} + \frac{1}{4!}\frac{h^4}{16} - \frac{1}{8}h^{\alpha\beta}h_{\beta\gamma}h^{\gamma\nu}h_{\nu\alpha}\right]\right\} \end{aligned} \quad (5.5)$$

Here the terms  $L^{(1)}$  and  $L^{(2)}$  without  $\Lambda$  are the same as in the ordinary massless gravity:

$$\begin{aligned} L^{(1)} &\equiv -\frac{1}{4}h^{\alpha\beta}(\partial_{\alpha}h)(\partial_{\beta}h) + \frac{1}{2}h^{\mu\nu}(\partial_{\mu}h^{\alpha\beta})(\partial_{\nu}h^{\alpha\beta}) + h^{\alpha\beta}(\partial_{\nu}h^{\alpha\mu})(\partial_{\mu}h^{\beta\nu}) \\ &+ \frac{1}{2}h^{\mu\nu}(\partial_{\alpha}h^{\mu\nu})(\partial_{\alpha}h) - h^{\mu\nu}(\partial_{\rho}h^{\alpha\mu})(\partial^{\rho}h^{\alpha\nu}) \end{aligned} \quad (5.6)$$

$$\begin{aligned} L^{(2)} &\equiv -h_{\alpha\rho}h_{\beta}^{\rho}(\partial_{\nu}h^{\alpha\mu})(\partial_{\mu}h^{\beta\nu}) - \frac{1}{2}h_{\rho\beta}h_{\gamma}^{\beta}(\partial_{\alpha}h^{\rho\gamma})(\partial^{\alpha}h) \\ &- \frac{1}{4}h_{\mu\nu}(\partial_{\alpha}h^{\mu\nu})h_{\rho\gamma}(\partial^{\alpha}h^{\rho\gamma}) + \frac{1}{2}h_{\mu\nu}(\partial_{\alpha}h^{\mu\nu})h^{\alpha\beta}(\partial_{\beta}h) - h_{\alpha\rho}(\partial_{\mu}h_{\gamma}^{\rho})(\partial_{\nu}h^{\alpha\gamma})h^{\mu\nu} \\ &+ h_{\rho\beta}h_{\gamma}^{\beta}(\partial_{\mu}h^{\alpha\rho})(\partial_{\alpha}h_{\gamma}^{\mu}) + \frac{1}{2}h_{\alpha\rho}h_{\beta\gamma}(\partial_{\mu}h^{\alpha\gamma})(\partial^{\mu}h^{\beta\rho}). \end{aligned} \quad (5.7)$$

The terms  $O(\kappa^{-2})$  and  $O(\kappa^{-1})$  in the Lagrangian (5.5) are irrelevant because they give no equation of motion. The term  $O(\kappa^0)$  and quadratic in  $h$

$$L^{(0)} \equiv \frac{1}{2}(\partial^\mu h^{\alpha\beta})(\partial_\mu h_{\alpha\beta}) - (\partial_\mu h^{\alpha\beta})(\partial_\beta h^\mu_\alpha) - \frac{1}{4}(\partial_\alpha h)(\partial^\alpha h) + \Lambda\left(\frac{h^2}{2} - h^{\alpha\beta}h_{\alpha\beta}\right) \quad (5.8)$$

gives the following Euler-Lagrange equation

$$\partial^2 h^{\alpha\beta} - \partial_\mu(\partial^\beta h^{\alpha\mu} + \partial^\alpha h^{\beta\mu}) - \frac{1}{2}\eta^{\alpha\beta}\partial^2 h - \Lambda(\eta^{\alpha\beta}h - 2h^{\alpha\beta}) = 0. \quad (5.9)$$

Taking the trace we find

$$\partial_\mu\partial_\alpha h^{\alpha\mu} = -\frac{1}{2}\partial^2 h - \Lambda h. \quad (5.10)$$

Differentiating (5.9) by  $\partial_\alpha$  and substituting (5.10) we derive the Hilbert gauge condition

$$\partial_\alpha h^{\alpha\beta} = 0. \quad (5.11)$$

Then (5.10) reduces to the Klein-Gordon equation

$$\partial^2 h + 2\Lambda h = 0 \quad (5.12)$$

and from (5.9) we obtain the Klein-Gordon equation for the tensor field

$$\partial^2 h^{\alpha\beta} + 2\Lambda h^{\alpha\beta} = 0 \quad (5.13)$$

This means the graviton becomes massive with mass

$$m^2 = 2\Lambda. \quad (5.14)$$

Taking the current value of  $\Lambda$  one finds a tiny mass ( $\approx 10^{-32}$  eV) for the graviton, so that our massive theory passes all direct tests of general relativity. However, gravitational radiation requires a thorough investigation.

The cubic part  $O(\kappa^1)$  in (5.5) gives the coupling. Using (5.14) this agrees exactly with the pure graviton coupling terms in (3.61), if we multiply with an overall factor  $-4$ . Similarly, the pure graviton couplings in the quartic part  $O(\kappa^2)$  agree with (4.12) without the ghost terms, if we multiply by 16. This shows that our *massive gravity is the quantum gauge theory corresponding to classical gravity with a cosmological term*. Of course, the ghost couplings cannot be derived from the simple Lagrangian above. To guess the right Lagrangian with ghost fields (fermionic and bosonic) is not quite straightforward. The best way to deduce the full theory is quantum gauge invariance, instead of some classical heuristics.

In spin 1 gauge theories with massive gauge fields a Higgs field is necessary to "generate" the masses. In the framework of quantum gauge theory it is simply needed to restore gauge invariance to second order [29]; first order gauge invariance holds without the Higgs couplings. It was a surprise for us when we found that massive gravity is gauge invariant to first *and*



second order without a Higgs field. In a way this is even disappointing because a gravitational Higgs field would be a nice candidate for the non-barionic dark matter.

A last remark is concerned with the mass zero limit of our theory. As noticed above, in the limit  $m \rightarrow 0$  the bosonic ghost  $v^\mu$  does not completely decouple from the graviton. In fact, the term  $4h^{\mu\nu}(\partial_\mu v_\rho)(\partial_\nu v^\rho)$  survives in (3.61). That means the resulting massless theory is not identical with usual quantum gravity as discussed in [29], for example. This leads to the conclusion that there exists at least two different quantum gauge theories which correspond to classical (massless) general relativity, one with an additional Bose field  $v^\mu$  and one without.

## References

- [1] A. Ashtekar, “*Lectures on non Perturbative Canonical Gravity*”, World Scientific, 1991
- [2] N. M. Arkani-Hamad, H. Georgi, M. D. Schwartz, “*Effective Field Theory for Massive Gravitons and Gravity in Theory Space*”, hep-th/0210184
- [3] D. M. Capper, G. Leibbrandt, M. Ramón Medrano, “*Calculation of the Graviton Self-Energy Using Dimensional Regularization*”, Phys. Rev. **D 8** (1973) 4320-4331
- [4] J. F. Donoghue, “*General Relativity as an Effective Field Theory: The Leading Quantum Corrections*”, Phys. Rev. **D 6** (1994) 3874-3888
- [5] H. van Dam, M. Veltman, “*Massive and Massless Yang-Mills and Gravitational Fields*”, Nucl. Phys. **B 22** (1970) 397-411
- [6] T. Damour, I. I. Kogan, A. Papazoglou, “*Spherically Symmetric Spacetimes in massive Gravity*”, Phys. Rev. **D 67** (2003) 064009:1-25
- [7] H. Epstein, V. Glaser, “*The Rôle of Locality in Perturbation Theory*”, Ann. Inst. H. Poincaré **19 A** (1973) 211-295
- [8] R. P. Feynman, “*Quantum Theory of Gravitation*”, Acta Phys. Polonica **XXIV** (1963) 679-722
- [9] V. Glaser, “*Electrodynamique Quantique*”, L’enseignement du 3e cycle de la physique en Suisse Romande (CICP), Semestre d’hiver 1972/73
- [10] J. N. Goldberg, “*Conservation Laws in General Relativity*”, Phys. Rev. **111** (1958) 315-320
- [11] D. R. Grigore, “*On the Uniqueness of the Non-Abelian Gauge Theories in the Epstein-Glaser Approach to Renormalization Theory*”, hep-th/9806244, Romanian Journ. Phys. **44** (1999) 853-913
- [12] D. R. Grigore, “*The Standard Model and its Generalisations in Epstein-Glaser Approach to Renormalisation Theory*”, hep-th/9810078, Journ. Phys. **A 33** (2000) 8443-8476
- [13] D. R. Grigore, “*On the Quantization of the Linearized Gravitational Field*”, hep-th/9905190, Class. Quant. Grav. **17** (2000) 319-344
- [14] N. Grillo, “*Some Aspects of Quantum Gravity in the Causal Approach*”, hep-th/9903011, the 4th workshop on “Quantum Field Theory under the Influence of External Conditions”, Leipzig, Germany, Sept. 1998
- [15] N. Grillo, “*Finite One-Loop Calculations in Quantum Gravity: Graviton Self-Energy, Perturbative Gauge Invariance and Slavnov-Ward Identities*”, hep-th/9912097

- [16] N. Grillo, “*Finite One-Loop Corrections and Perturbative Gauge Invariance in Quantum Gravity Coupled to Photon Fields*”, hep-th/9912114
- [17] N. Grillo, “*Scalar Matter Coupled to Quantum Gravity in the Causal Approach: Finite One-Loop Calculations and Perturbative Gauge Invariance*”, hep-th/9912128
- [18] S. N. Gupta, “*Einstein’s Other Theories of Gravitation*”, Rev. Mod. Phys. **29** (1957) 334-336
- [19] S. N. Gupta, “*Supplementary Conditions in the Quantized Gravitational Theory*”, Phys. Rev. **172** (1968) 1303-1307
- [20] M. H. Goroff, A. Sagnotti, “*The Ultraviolet Behaviour of Einstein Gravity*”, Nucl. Phys. **B 266** (1986) 709-736
- [21] G. Gabadadze, M. Shifman, “*Softly Massive Gravity*”, hep-th/0312289
- [22] G. ’t Hooft, M. Veltman, “*One-Loop Divergences in the Theory of Gravitation*”, Ann. Inst. H. Poincaré **XX** (1974) 69-94
- [23] T. Kugo, I. Ojima, “*Subsidiary Conditions and Physical S-Matrix Unitarity in Indefinite-Metric Quantum Gravitation Theory*”, Nucl. Phys. **B 144** (1978) 234- 252
- [24] T. Kugo, I. Ojima, “*Local Covariant Operator Formalism of Non-Abelian Gauge Theories and Quark Confinement Problem*”, Suppl. of Progr. of Theor. Phys. **66** (1979) 1-130
- [25] M. A. Luzy, M. Porrati, R. Rattazzi, “*Strong Interactions and Stability in the DGP Model*”, hep-th/0303116
- [26] V. I. Ogievetsky, I. V. Polubarinov, “*Interacting Field of Spin 2 and the Einstein Equations*”, Ann. Phys. **35** (1965) 167-208
- [27] R. Rattazzi, G. A. Scrucca, A. Struma, “*Brane to Brane Gravity Mediation and Supersymmetry Breaking*”, hep-th/0305184
- [28] G. Scharf, “*Finite Quantum Electrodynamics: The Causal Approach*”, (second edition) Springer, 1995
- [29] G. Scharf, “*Quantum Gauge Theories: A True Ghost Story*”, J. Wiley, 2001
- [30] I. Schorn, “*Gauge Invariance of Quantum Gravity in the Causal Approach*”, Clas. Quantum Grav. **14** (1997) 653-669
- [31] G. Scharf, M. Wellmann, “*Quantum Gravity from Perturbative Gauge Invariance*”, Gen. Rel. Gravit. **33** (2001) 553, hep-th/9903055
- [32] T. Thiemann, “*Lectures on Loop Quantum Gravity*”, gr-qc/0210094

- [33] W. Thirring, “*An Alternative Approach to the Theory of Gravitation*”, Ann. Phys. **16** (1961) 96-117
- [34] A. I. Vainshtein, “*To the Problem of Nonvanishing Gravitation Mass*”, Phys. Lett. **39 B** (1972) 393-394
- [35] A. E. M. van de Ven, “*Two-Loop Quantum Gravity*”, Nucl. Phys. **B 378** (1992) 309-366
- [36] S. A. A. Zaidi, “*Self-Energy of the Graviton in Second Order*”, J. Phys. A: Math. Gen **24** (1991) 4325-4334
- [37] S. Weinberg, “*The Quantum Theory of Fields*”, vol. 1 and 2, Cambridge Univ. Press, 1995